Discrete Applied Mathematics 157 (2009) 177–182

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Offensive r-alliances in graphs

Henning Fernau [a], Juan A. Rodríguez [b,*], José M. Sigarreta [c]

[a] FB 4-Abteilung Informatik, Universität Trier, 54286 Trier, Germany

[b] Departament d'Enginyeria Informàtica i Matemàtiques, Universitat Rovira i Virgili, Av. Països Catalans 26, 43007 Tarragona, Spain

[c] Departamento de Matemáticas, Universidad Autónoma Popular del Estado de Puebla, 21 Sur 1103, Col. Santiago, Puebla, Mexico

ARTICLE INFO

Article history:
Received 25 March 2007
Received in revised form 22 May 2008
Accepted 3 June 2008
Available online 14 July 2008

Keywords:
Computational complexity
Offensive alliances
Alliances in graphs
Domination

ABSTRACT

Let $G = (V, E)$ be a simple graph. For a nonempty set $X \subset V$, and a vertex $v \in V$, $\delta_X(v)$ denotes the number of neighbors v has in X. A nonempty set $S \subset V$ is an *offensive r-alliance* in G if $\delta_S(v) \geq \delta_{\bar{S}}(v) + r$, $\forall v \in \partial(S)$, where $\partial(S)$ denotes the boundary of S. An offensive r-alliance S is called *global* if it forms a dominating set. The *global offensive r-alliance number* of G, denoted by $\gamma_r^o(G)$, is the minimum cardinality of a global offensive r-alliance in G. We show that the problem of finding optimal (global) offensive r-alliances is NP-complete and we obtain several tight bounds on $\gamma_r^o(G)$.

1. Introduction

The mathematical properties of alliances in graphs were first studied by Kristiansen, Hedetniemi and Hedetniemi [12]. They proposed different types of alliances: namely, defensive alliances [10–12,19], offensive alliances [4,6,16,20] and dual alliances or powerful alliances [1]. A generalization of these alliances called r-alliances was presented by Shafique and Dutton [17,18].

In this paper, we study the mathematical properties of offensive r-alliances. We begin by stating the terminology used. Throughout this article, $G = (V, E)$ denotes a simple graph of order $|V| = n$. We denote two adjacent vertices u and v by $u \sim v$. For a nonempty set $X \subseteq V$, and a vertex $v \in V$, $N_X(v)$ denotes the set of neighbors v has in X: $N_X(v) := \{u \in X : u \sim v\}$, and the degree of v in X will be denoted by $\delta_X(v) = |N_X(v)|$. We denote the degree of a vertex $v \in V$ by $d(v)$ and the degree sequence of G by $d_1 \geq d_2 \geq \cdots \geq d_n = \delta$. The complement of the vertex-set S in V is denoted by $\bar{S}$ and the boundary, $\partial(S)$, of S is defined by

$$\partial(S) := \bigcup_{v \in S} N_{\bar{S}}(v).$$

For $r \in \{2 - d_1, \ldots, d_1\}$, a nonempty set $S \subset V$ is an *offensive r-alliance* in G if for every $v \in \partial(S)$,

$$\delta_S(v) \geq \delta_{\bar{S}}(v) + r \qquad (1)$$

or, equivalently,

$$\delta(v) \geq 2\delta_{\bar{S}}(v) + r. \qquad (2)$$

An offensive 1-alliance is an *offensive alliance* and an offensive 2-alliance is a *strong offensive alliance* as defined in [6,16,20].

* Corresponding author.
E-mail addresses: fernau@uni-trier.de (H. Fernau), juanalberto.rodriguez@urv.cat (J.A. Rodríguez), josemaria.sigarreta@upaep.mx (J.M. Sigarreta).

0166-218X/$ – see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.dam.2008.06.001

The *offensive r-alliance number* of G, denoted by $a_r^o(G)$, is defined as the minimum cardinality of an offensive r-alliance in G. Notice that

$$a_{r+1}^o(G) \geq a_r^o(G). \tag{3}$$

The offensive 1-alliance number of G is known as the *offensive alliance number* of G and the offensive 2-alliance number is known as the *strong offensive alliance number* [6,16,20].

A set $S \subset V$ is a *dominating set* in $G = (V, E)$ if for every vertex $u \in \bar{S}$, $\delta_S(u) > 0$ (every vertex in $\bar{S}$ is adjacent to at least one vertex in S). The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set in G.

An offensive r-alliance S is called *global* if it forms a dominating set, i.e., $\partial(S) = \bar{S}$. The *global offensive r-alliance number* of G, denoted by $\gamma_r^o(G)$, is the minimum cardinality of a global offensive r-alliance in G. Clearly,

$$\gamma_{r+1}^o(G) \geq \gamma_r^o(G) \geq \gamma(G) \quad \text{and} \quad \gamma_r^o(G) \geq a_r^o(G). \tag{4}$$

Notice that if every vertex of G has even degree and r is odd, $r = 2l - 1$, then every offensive $(2l - 1)$-alliance in G is an offensive $(2l)$-alliance. Hence, in such a case, $a_{2l-1}^o(G) = a_{2l}(G)$ and $\gamma_{2l-1}^o(G) = \gamma_{2l}^o(G)$. Analogously, if every vertex of G has odd degree and r is even, $r = 2l$, then every offensive $(2l)$-alliance in G is an offensive $(2l + 1)$-alliance. Hence, in such a case, $a_{2l}^o(G) = a_{2l+1}^o(G)$ and $\gamma_{2l}^o(G) = \gamma_{2l+1}^o(G)$.

2. On the complexity of finding optimal offensive *r*-alliances

For the class of complete graphs of order n, $G = K_n$, we have the exact value of $a_r^o(G)$. That is,

$$\begin{aligned} n - 1 = \;& a_{n-1}^o(K_n) = a_{n-2}^o(K_n) \\ \geq \;& a_{n-3}^o(K_n) = a_{n-4}^o(K_n) = n - 2 \\ & \cdots \\ \geq \;& a_{5-n}^o(K_n) = a_{4-n}^o(K_n) = 2 \\ \geq \;& a_{3-n}^o(K_n) = 1. \end{aligned}$$

Hence, for every $r \in \{3 - n, \ldots, n - 1\}$, $a_r^o(K_n) = \left\lceil \frac{n+r-1}{2} \right\rceil$. In this case, every offensive r-alliance is global and every vertex-set of cardinality $\left\lceil \frac{n+r-1}{2} \right\rceil$ is a (global) offensive r-alliance.

As we will see below, in general, the problem of finding optimal (global) offensive r-alliances is NP-complete. That is, we are interested in the computational complexity of the following optimization problems.

Offensive r-Alliance problem (r-**OA**):

Given: A graph $G = (V, E)$ and a positive integer $k \leq |V|$.
Question: Is there an offensive r-alliance in G of size k or less?

Global offensive r-Alliance problem (r-**GOA**):

Given: A graph $G = (V, E)$ and a positive integer $k \leq |V|$.
Question: Is there a global offensive r-alliance in G of size k or less?

2.1. Offensive alliances

Our reasoning will use and generalize the following observation:

Proposition 1 (*[6]*). *On cubic graphs, every vertex cover is a strong offensive alliance and vice versa.*

With some gadgetry, this observation was used in [8] to show NP-hardness of finding small strong offensive alliances. We will generalize those results in the following.

Theorem 2. *$\forall r$: r-OA is NP-complete.*

Proof. For each r, we have to show that r-OA belongs to NP and that r-OA is NP-hard.

(A) It is easy to verify that a given vertex set forms an r-OA. Therefore, a nondeterministic Turing machine running in polynomial time can first guess at most k vertices and then test if that vertex set if a valid r-OA. Hence, r-OA is in NP.

(B) We first show NP-hardness in the case that $r \geq 3$. For any connected r-regular graph $G = (V, E)$, it can be seen that $C \subseteq V$ is a minimum vertex cover if C is a minimum r-offensive alliance. Clearly, any vertex cover is an r-OA. Let S be an r-OA. If $S = V$, then the claim is true. Otherwise, discuss $x \in \partial(S)$. Since S is an r-OA and G is r-regular, all neighbors of x are in S (*). If there were an edge $e = \{u, v\}$ with $u, v \in \bar{S}$, then, since $S \neq \emptyset$ (by definition) and G is connected, there exists a path p from u to some $y \in S$. On p, we must find some $x \in \partial(S)$ that has a neighbor in $\bar{S}$, contradicting (*). Hence, no such edge e exists, which means that S forms a vertex cover. Since it is well-known that the vertex cover problem, restricted to r-regular graphs is NP-complete for any $r \geq 3$, see [7] for a recent account related to approximability results, the claim follows for $r \geq 3$.

In fact, the argument corresponding to the special case $r = 3$ is also valid for strong offensive alliances, and this is exactly the mentioned result from [6,8].

(C) Now, we show NP-hardness for the remaining cases. More specifically, we prove: if r-OA is NP-hard and $r \leq 2$, then so is $(r-1)$-OA. By induction, the whole claim will follow.

Let $(G = (V, E), k)$ be an instance of r-OA, with $n = |V|$. We construct an instance of $(r-1)$-OA as follows: $G' = (V', E')$ with $V' = V \times \{1, 2, 3\} \cup \{c_1, \ldots, c_\ell\}$. In E', we find the following edges (and only those):
– $\{(u, 1), (v, 1)\} \in E'$ if $\{(u, 2), (v, 2)\} \in E'$ iff $\{u, v\} \in E$;
– $\{(u, 1), (u, 3)\} \in E'$ and $\{(u, 2), (u, 3)\} \in E'$ for any $u \in V$;
– $\{(u, 3), c_j\} \in E'$ for any $u \in V$ and any $1 \leq j \leq 3 - r$;
– $\{c_i, c_j\} \in E'$ for any $1 \leq i < j \leq \ell$, i.e., $C = \{c_1, \ldots, c_\ell\}$ forms a clique.
Let $k' = 2k$ and $\ell = 3n + 3 - r$. As in [8], one can show that S is an r-OA of size at most k for G iff $S \times \{1, 2\}$ is a $(r-1)$-OA of size at most k' for G', and that there is no other possibility to form smaller $(r-1)$-OAs in G' due to the attached clique C. The decisive observation is that $C \cap \partial(S') = \emptyset$ for any valid $(r-1)$-OA S' in G' of size at most k', because there would be far too many neighbors of $c \in C \cap \partial(S')$ that are not in S', since $|S'| \leq k'$. This observation implies that no vertex from S' lies in $V \times \{3\}$ (nor in C). Hence, all vertices from S' are to be found in $V \times \{1, 2\}$, which induce two copies of G. Discuss $(u, 1) \in S'$ (the case $(u, 2) \in S'$ being symmetric). Hence, $(u, 3) \in \partial(S')$. $(u, 3)$ has $3 - r$ neighbors in $C \subseteq \bar{S}'$. In order to satisfy $\delta_{S'}((u, 3)) \geq \delta_{\bar{S}'}((u, 3)) + (r-1) = 2$, $(u, 2) \in S'$ is necessarily true. Since $S' \neq \emptyset$, $|S'| \leq k'$, the projection of S' on the first component entails a subset S of V that forms an r-OA in G.
The converse is seen much easier: If S is an r-OA in G, then $S \times \{1, 2\}$ forms an $(r-1)$-OA in G'. □

The proof of the preceding theorem allows the following sharpened formulation, relying on known NP-hardness results for vertex cover:

Corollary 3. *For $r > 1$, r-OA is NP-hard, even when restricted to r-regular planar graphs.*

2.2. Global offensive alliances

Cami et al. [2] showed NP-completeness for $r = 1$. We are going to modify their construction to show NP-completeness for any fixed r. Since we are dealing with the degree of vertices both in G and within the new graph G' as constructed below, we are going to attach G and G' to δ to avoid confusion in our notation.

Theorem 4. *$\forall r$: r-GOA is NP-complete.*

Proof. Membership in NP is seen similar to the previous theorem.

The construction in [2] can be modified to work for any case $r \leq 1$. Let (G, k) be an instance of Dominating Set with minimum degree $|r| + 1$, with $G = (V, E)$. To any $v \in V$, attach $\delta_G(v) + r - 1 \geq 0$ copies of K_2 with one edge per K_2-copy, this way yielding a new graph $G' = (V', E')$ with G as a subgraph; call the new neighbors of vertices from V A-vertices and collect them into set A, and call $N(A) \setminus V$ B-vertices.

If $D \subseteq V$ is a dominating set in G, then $S = D \cup A$ is a r-GOA. Clearly, S is a dominating set in G'. Now, consider a B-vertex v. Obviously, $N(v) \subseteq A$, and therefore $|N_{G'}(v) \cap S| \geq |N_{G'}(v) \cap \bar{S}| + r$. Any vertex $v \in V \setminus D$ has a neighbor $d \in D$. Hence, $|N_{G'}(v) \cap \bar{S}| \leq \delta_G(v) - 1$, while $|N_{G'}(v) \cap S| \geq \delta_G(v) + (r-1) + 1 = \delta_G(v) + r$. Therefore, S is a valid r-GOA.

Conversely, let S be a r-GOA of G'. Since S is a dominating set, for each K_2-copy attached to G, either the corresponding A- or the corresponding B-vertex is in S. Consider some $v \in V \setminus S$. v must be dominated. If no neighbor of v in V is in S, then $|N_{G'}(v) \cap S| \leq \delta_G(v) + r - 1$, while $|N_{G'} \cap \bar{S}| \geq \delta_G(v)$, which leads to a contradiction. Hence, $S \cap V$ is a dominating set in G.

Combining the arguments, we obtain: $G = (V, E)$ has a dominating set of size at most k if $G' = (V', E')$ has a r-GOA of size $k + \sum_v (\delta_G(v) + r - 1) = k + (r-1)|V| + 2|E|$.

Now, we consider the case $r \geq 2$. Let (G, k) be an instance of Dominating Set with minimum degree 1, with $G = (V, E)$. To any $v \in V$, attach $\delta_G(v) + r - 1 \geq 1$ so-called A-vertices. All A-vertices together form an independent set. Let $A(v) = \{(v, 1), \ldots, (v, \delta_G(v) + r - 1)\}$ denote the set of A-vertices attached to $v \in V$. We denote the B-vertices attached to the A-vertices in $A(v)$ by $B(v)$ and can describe them as $B(v) = \binom{A(v)}{r}$, i.e., the r-element subsets of $A(v)$. Each $X \in B(v)$ has as neighbors exactly the A-vertices listed in X. This describes the graph $G' = (V', E')$ as obtained from G.

If $D \subseteq V$ is a dominating set in G, then $S = D \cup A$ is a r-GOA in G'. Clearly, S is a dominating set in G'. Now, consider a B-vertex v. Obviously, $N(v) \subseteq A(v)$, and therefore $|N_{G'}(v) \cap S| = r \geq |N_{G'}(v) \cap \bar{S}| + r$. Any vertex $v \in V \setminus D$ has a neighbor $d \in D$. Hence, $|N_{G'}(v) \cap \bar{S}| \leq \delta_G(v) - 1$, while $|N_{G'}(v) \cap S| \geq \delta_G(v) + (r-1) + 1 = \delta_G(v) + r$. Therefore, S is a valid r-GOA.

Conversely, let S be a r-GOA of G' of size $k + |A|$. Notice that this bound is met if $S \cap V$ is a dominating set in G and all A-vertices go into S. Consider an $A(v)$-vertex x and assume $x \notin S$. Then, either there is a $y \in S \cap N(x) \cap B(v)$, or $v \in S$, since otherwise x would not be dominated. Altogether, x has $\binom{\delta_G(v) + r - 1}{r} + 1$many neighbors. Since S is an r-GOA, more than $|A(v)| = \delta_G(v) + r - 1$ vertices from the gadget attached to v would be in S, this way violating the bound on the size of S.

Consider some $v \in V \setminus S$. v must be dominated. If no neighbor of v in V is in S, then $|N_{G'}(v) \cap S| \leq \delta_G(v) + r - 1$, while $|N_{G'} \cap \bar{S}| \geq \delta_G(v)$, which leads to a contradiction. Hence, $S \cap V$ is a dominating set in G.

Combining the arguments, we obtain: $G = (V, E)$ has a dominating set of size at most k if $G' = (V', E')$ has a r-GOA of size $k + \sum_v (\delta_G(v) + r - 1) = k + (r - 1)|V| + 2|E|$. □

3. Bounding the offensive r-alliance number

Theorem 5. *For any graph G of order n and minimum degree δ, and for every $r \in \{2 - \delta, \ldots, \delta\}$,*

$$\left\lceil \frac{\delta + r}{2} \right\rceil \leq a_r^o(G) \leq \gamma_r^o(G) \leq n - \left\lceil \frac{\delta - r + 2}{2} \right\rceil.$$

Proof. Let v be a vertex of minimum degree in G and let $Y \subset N_V(v)$ such that $|Y| = \left\lceil \frac{\delta+r}{2} \right\rceil$. Let $S = \{v\} \cup N_V(v) - Y$. Hence, $\bar{S}$ is a dominating set and

$$\delta_{\bar{S}}(v) = \left\lceil \frac{\delta + r}{2} \right\rceil \geq \left\lfloor \frac{\delta + r}{2} \right\rfloor = \delta - \left\lceil \frac{\delta + r}{2} \right\rceil + r = \delta_S(v) + r.$$

Thus,

$$\delta_{\bar{S}}(u) \geq \delta_{\bar{S}}(v) \geq \delta_S(v) + r \geq \delta_S(u) + r, \quad \forall u \in S.$$

Therefore, $\bar{S}$ is a global offensive r-alliance in G and, as a consequence, the upper bound follows.

On the other hand, let $X \subset V$ be an offensive r-alliance in G. For every $v \in \partial(X)$ we have

$$d(v) = \delta_X(v) + \delta_{\bar{X}}(v)$$
$$d(v) \leq \delta_X(v) + \frac{d(v) - r}{2}$$
$$\frac{d(v) + r}{2} \leq \delta_X(v) \leq |X|$$
$$\frac{\delta + r}{2} \leq |X|.$$

Therefore, the lower bound follows. □

The bounds are attained for every r in the case of the complete graph $G = K_n$.

A set $S \subset V$ is a k-dominating set if for every $v \in \bar{S}$, $\delta_S(v) \geq k$. The *k-domination number* of G, $\gamma_k(G)$, is the minimum cardinality of a k-dominating set in G. The following result generalizes, to r alliances, some previous results obtained for $r = 1$ and $r = 2$ [14,16].

Theorem 6. *For any simple graph G of order n, minimum degree δ, and Laplacian spectral radius*[1] μ_*,

$$\left\lceil \frac{n}{\mu_*} \left\lceil \frac{\delta + r}{2} \right\rceil \right\rceil \leq \gamma_r^o(G) \leq \left\lfloor \frac{\gamma_r(G) + n}{2} \right\rfloor.$$

Proof. Let $H \subset V$ be an r-dominating set of G of minimum cardinality. If $|\bar{H}| = 1$, then $\gamma_r(G) = n - 1$ and $\gamma_r^o(G) \leq n - 1$. If $|\bar{H}| \neq 1$, let $\bar{H} = X \cup Y$ be a partition of $\bar{H}$ such that the edge-cut between X and Y has the maximum cardinality. Suppose $|X| \leq |Y|$. For every $v \in Y$, $\delta_H(v) \geq r$ and $\delta_X(v) \geq \delta_Y(v)$. Therefore, the set $W = H \cup X$ is a global offensive r-alliance in G, i.e., for every $v \in Y$, $\delta_W(v) \geq \delta_Y(v) + r$. Then we have,

$$2|X| + \gamma_r(G) \leq n \tag{5}$$

and

$$\gamma_r^o(G) \leq |X| + \gamma_r(G). \tag{6}$$

Thus, by (5) and (6), we obtain the upper bound.

[1] i.e., the largest Laplacian eigenvalue of G. The reader is referred to [5,13] for a detailed study and survey on the Laplacian matrix of a graph and its eigenvalues.

It was shown in [9] that the Laplacian spectral radius of G, μ_*, satisfies

$$\mu_* = 2n \max\left\{ \frac{\sum_{v_i \sim v_j} (w_i - w_j)^2}{\sum_{v_i \in V} \sum_{v_j \in V} (w_i - w_j)^2} : w \neq \alpha \mathbf{j} \text{ for } \alpha \in \mathbb{R} \right\}, \tag{7}$$

where $V = \{v_1, v_2, \ldots, v_n\}$, $\mathbf{j} = (1, 1, \ldots, 1)$ and $w \in \mathbb{R}^n$. Let $S \subset V$. From (7), taking $w \in \mathbb{R}^n$ defined as

$$w_i = \begin{cases} 1 & \text{if } v_i \in S; \\ 0 & \text{otherwise} \end{cases}$$

we obtain

$$\mu_* \geq \frac{n \sum_{v \in \bar{S}} \delta_S(v)}{|S|(n - |S|)}. \tag{8}$$

Moreover, if S is a global offensive r-alliance in G,

$$\delta_S(v) \geq \left\lceil \frac{d(v) + r}{2} \right\rceil, \quad \forall v \in \bar{S}. \tag{9}$$

Thus, (8) and (9) lead to

$$\mu_* \geq \frac{n}{|S|} \left\lceil \frac{\delta + r}{2} \right\rceil. \tag{10}$$

Therefore, solving (10) for $|S|$ we obtain the lower bound. □

The above-mentioned bounds are attained, for instance, in the case of the complete graph of order n.

Corollary 7. *For any simple graph G of order n, minimum degree δ, and for every $r \in \{1, \ldots, \delta\}$,*

$$\gamma_r^o(G) \leq \left\lfloor \frac{n(2r + 1)}{2r + 2} \right\rfloor.$$

Proof. The bound immediately follows from the following bound on $\gamma_r(G)$ [3]:

$$\delta \geq r \Rightarrow \gamma_r(G) \leq \frac{rn}{r + 1}. \quad \square \tag{11}$$

Corollary 8. *Let $\mathcal{L}(G)$ be the line graph of a δ-regular graph G of order n. Then*

$$\gamma_r^o(\mathcal{L}(G)) \geq \frac{n}{4} \left\lceil \frac{2(\delta - 1) + r}{2} \right\rceil.$$

Proof. We denote by A the adjacency matrix of $\mathcal{L}(G)$ and by $2(\delta - 1) = \lambda_0 > \lambda_1 > \cdots > \lambda_b = -2$ its distinct eigenvalues. We denote by L the Laplacian matrix of $\mathcal{L}(\Gamma)$ and by $\mu_0 = 0 < \mu_1 < \cdots < \mu_b$ its distinct Laplacian eigenvalues. Then, since $L = 2(\delta - 1)I_n - A$, the eigenvalues of both matrices, A and L, are related by

$$\mu_l = 2(\delta - 1) - \lambda_l, \quad l = 0, \ldots, b. \tag{12}$$

Thus, the Laplacian spectral radius of $\mathcal{L}(G)$ is $\mu_b = 2\delta$. Therefore, the result immediately follows. □

There are some immediate bounds on $\gamma_r^o(G)$ derived from the following remarks.

Remark 9. If S is an independent set in G, then $\bar{S}$ is a global offensive r-alliance in G ($r \leq \delta$).

Remark 10. All global offensive r-alliance in G is a $\lceil \frac{\delta+r}{2} \rceil$-dominating set in G ($r \geq 2 - \delta$).

Therefore, for $2 - \delta \leq r \leq \delta$, the following bounds follow.

$$\gamma_{\lceil \frac{\delta+r}{2} \rceil}(G) \leq \gamma_r^o(G) \leq n - \alpha(G), \tag{13}$$

where $\alpha(G)$ denotes the independence number of G.

The reader is referred to our previous works [14,15,20,16] for a more detailed study on offensive 1-alliances and offensive 2-alliances.

References

[1] R. Brigham, R. Dutton, S. Hedetniemi, A sharp lower bound on the powerful alliance number of $C_m \times C_n$, Congressus Numerantium 167 (2004) 57–63.
[2] A. Cami, H. Balakrishnan, N. Deo, R. Dutton, On the complexity of finding optimal global alliances, Journal of Combinatorial Mathematics and Combinatorial Computing 58 (2006).
[3] E.J. Cockayne, B. Gamble, B. Shepherd, An upper bound for the k-domination number of a graph, Journal of Graph Theory 9 (4) (1985) 533–534.
[4] M. Chellali, T.W. Haynes, Global alliances and independence in trees, Discussiones Mathematicae Graph Theory 27 (1) (2007) 19–27.
[5] D.M. Cvetković, M. Doob, H. Sachs, Spectra of Graphs - Theory and Application, Academic Press, New York, 1980.
[6] O. Favaron, G. Fricke, W. Goddard, S. Hedetniemi, S.T. Hedetniemi, P. Kristiansen, R.C. Laskar, R.D. Skaggs, Offensive alliances in graphs, Discussiones Mathematicae Graph Theory 24 (2) (2004) 263–275.
[7] U. Feige, Vertex cover is hardest to approximate on regular graphs, Technical Report MCS 03-15 of the Weizmann Institute, 2003.
[8] H. Fernau, D. Raible, Alliances in graphs: A complexity-theoretic study, Software Seminar SOFSEM 2007, Student Research Forum, Proceedings Vol. II, 61–70.
[9] M. Fiedler, A property of eigenvectors of nonnegative symmetric matrices and its application to graph theory, Czechoslovak Mathematical Journal 25 (100) (1975) 619–633.
[10] G.H. Fricke, L.M. Lawson, T.W. Haynes, S.M. Hedetniemi, S.T. Hedetniemi, A note on defensive alliances in Graphs, Bulletin of the Institute of Combinatorics and its Applications 38 (2003) 37–41.
[11] T.W. Haynes, S.T. Hedetniemi, M.A. Henning, Global defensive alliances in graphs, The Electronic Journal of Combinatorics 10 (2003) 139–146.
[12] P. Kristiansen, S.M. Hedetniemi, S.T. Hedetniemi, Alliances in graphs, Journal of Combinatorial Mathematics and Combinatorial Computing 48 (2004) 157–177.
[13] B. Mohar, The Laplacian spectrum of graphs, in: Y. Alavi, G. Chartrand, O. Ollermann, A. Schwenk (Eds.), Graph Theory, Combinatorics, and Applications, John Wiley and Sons, Inc, New York, 1991, pp. 871–898.
[14] J.A. Rodríguez, J.M. Sigarreta, Spectral study of alliances in graphs, Discussiones Mathematicae Graph Theory 27 (1) (2007) 143–157.
[15] J.A. Rodríguez, J.M. Sigarreta, Global alliances in planar graphs, AKCE–International Journal of Graphs and Combinatorics 4 (1) (2007) 83–98.
[16] J.A. Rodríguez-Velázquez, J.M. Sigarreta, Global offensive alliances in graphs, Electronic Notes in Discrete Mathematics 25 (2006) 157–164.
[17] K.H. Shafique, R.D. Dutton, Maximum alliance-free and minimum alliance-cover sets, Congressus Numerantium 162 (2003) 139–146.
[18] K.H. Shafique, R. Dutton, A tight bound on the cardinalities of maximum alliance-free and minimum alliance-cover sets, Journal of Combinatorial Mathematics and Combinatorial Computing 56 (2006) 139–145.
[19] J.M. Sigarreta, J.A. Rodríguez, On defensive alliance and line graphs, Applied Mathematics Letters 19 (12) (2006) 1345–1350.
[20] J.M. Sigarreta, J.A. Rodríguez, On the global offensive alliance number of a graph, Discrete Applied Mathematics, in press (doi:10.1016/j.dam.2008.02.007).