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Defensive k-alliances in graphs

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ABSTRACT

Let $G=(V,E)$ be a simple graph of order n and degree sequence $\delta_1\ge\delta_2\ge\cdots\ge\delta_n$. For a nonempty set $X\subseteq V$, and a vertex $v\in V$, $\delta_X(v)$ denotes the number of neighbors that v has in X. A nonempty set $S\subseteq V$ is a *defensive k-alliance* in $G=(V,E)$ if $\delta_S(v)\ge\delta_{\bar{S}}(v)+k$, $\forall v\in S$. The defensive k-alliance number of G, denoted by $a_k(G)$, is defined as the minimum cardinality of a defensive k-alliance in G. We study the mathematical properties of $a_k(G)$. We show that $\left\lceil\frac{\delta_n+k+2}{2}\right\rceil\le a_k(G)\le n-\left\lfloor\frac{\delta_n-k}{2}\right\rfloor$ and $a_k(G)\ge\left\lceil\frac{n(\mu+k+1)}{n+\mu}\right\rceil$, where μ is the algebraic connectivity of G and $k\in\{-\delta_n,\ldots,\delta_1\}$. Moreover, we show that for every $k,r\in\mathbb{Z}$ such that $-\delta_n\le k\le\delta_1$ and $0\le r\le\frac{k+\delta_n}{2}$, $a_{k-2r}(G)+r\le a_k(G)$ and, as a consequence, we show that for every $k\in\{-\delta_n,\ldots,0\}$, $a_k(G)\le\left\lceil\frac{n+k+1}{2}\right\rceil$. In the case of the line graph $\mathcal{L}(G)$ of a simple graph G, we obtain bounds on $a_k(\mathcal{L}(G))$ and, as a consequence of the study, we show that for any δ-regular graph, $\delta>0$, and for every $k\in\{2(1-\delta),\ldots,0\}$, $a_k(\mathcal{L}(G))=\delta+\left\lceil\frac{k}{2}\right\rceil$. Moreover, for any (δ_1,δ_2)-semiregular bipartite graph G, $\delta_1>\delta_2$, and for every $k\in\{2-\delta_1-\delta_2,\ldots,\delta_1-\delta_2\}$, $a_k(\mathcal{L}(G))=\left\lceil\frac{\delta_1+\delta_2+k}{2}\right\rceil$.

1. Introduction

The mathematical properties of alliances in graphs were first studied by Kristiansen, Hedetniemi and Hedetniemi [7]. They proposed different types of alliances: namely, defensive alliances [5–7,11], offensive alliances [2,9,10] and dual alliances or powerful alliances [1]. A generalization of these alliances called k-alliances was presented by Shafique and Dutton [12,13].

In this work, we study the mathematical properties of defensive k-alliances. We begin by stating the terminology used. Throughout this article, $G=(V,E)$ denotes a simple graph of order $|V|=n$ and size $|E|=m$. We denote two adjacent vertices u and v by $u\sim v$. For a nonempty set $X\subseteq V$, and a vertex $v\in V$, $N_X(v)$ denotes the set of neighbors that v has in X: $N_X(v):=\{u\in X: u\sim v\}$ and the degree of v in X will be denoted by $\delta_X(v)=|N_X(v)|$. We denote the degree of a vertex $v_i\in V$ by $\delta(v_i)$ (or by δ_i for short) and the degree sequence of G by $\delta_1\ge\delta_2\ge\cdots\ge\delta_n$. The subgraph induced by $S\subset V$ will be denoted by $\langle S\rangle$ and the complement of the set S in V will be denoted by $\bar{S}$.

A nonempty set $S\subseteq V$ is a *defensive k-alliance* in $G=(V,E)$, $k\in\{-\delta_1,\ldots,\delta_1\}$, if for every $v\in S$,

$$\delta_S(v)\ge\delta_{\bar{S}}(v)+k. \qquad (1)$$

A vertex $v\in S$ is said to be *k-satisfied* by the set S if (1) holds. Notice that (1) is equivalent to

$$\delta(v)\ge 2\delta_{\bar{S}}(v)+k. \qquad (2)$$

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A defensive (-1)-alliance is a *defensive alliance* and a defensive 0-alliance is a *strong defensive alliance* as defined in [7]. A defensive 0-alliance is also known as a *cohesive set* [14].

Defensive alliances are the mathematical model of web communities. Adopting the definition of Web community proposed recently by Flake, Lawrence, and Giles [3], "a Web community is a set of web pages having more hyperlinks (in either direction) to members of the set than to non-members".

2. Defensive k-alliance number

The *defensive k-alliance number* of G, denoted by $a_k(G)$, is defined as the minimum cardinality of a defensive k-alliance in G. Notice that

$$a_{k+1}(G) \geq a_k(G). \tag{3}$$

The defensive (-1)-alliance number of G is known as the *alliance number* of G and the defensive 0-alliance number is known as the *strong alliance number* [7,5,6]. For instance, in the case of the 3-cube graph, $G = Q_3$, every set composed by two adjacent vertices is a defensive alliance of minimum cardinality and every set composed by four vertices whose induced subgraph is isomorphic to the cycle C_4 is a strong defensive alliance of minimum cardinality. Thus, $a_{-1}(Q_3) = 2$ and $a_0(Q_3) = 4$.

Notice that if every vertex of G has even degree and k is odd, $k = 2l - 1$, then every defensive $(2l - 1)$-alliance in G is a defensive $(2l)$-alliance. Hence, in such a case, $a_{2l-1}(G) = a_{2l}(G)$. Analogously, if every vertex of G has odd degree and k is even, $k = 2l$, then every defensive $(2l)$-alliance in G is a defensive $(2l + 1)$-alliance. Hence, in such a case, $a_{2l}(G) = a_{2l+1}(G)$.

For some graphs, there are some values of $k \in \{-\delta_1, \dots, \delta_1\}$ such that defensive k-alliances do not exist. For instance, for $k \geq 2$ in the case of the star graph S_n, defensive k-alliances do not exist. By (2) we conclude that, in any graph, there are defensive k-alliances for all $k \in \{-\delta_1, \dots, \delta_n\}$. For instance, a defensive (δ_n)-alliance in $G = (V, E)$ is V. Moreover, if $v \in V$ is a vertex of minimum degree, $\delta(v) = \delta_n$, then $S = \{v\}$ is a defensive k-alliance for every $k \leq -\delta_n$. As $a_k(G) = 1$ for $k \leq -\delta_n$, hereafter we will only consider the cases $-\delta_n \leq k \leq \delta_1$. Moreover, the bounds shown in this work on $a_k(G)$, for $\delta_n \leq k \leq \delta_1$, are obtained by supposing that the graph G contains defensive k-alliances.

It was shown in [7] that for any graph G of order n and minimum degree δ_n,

$$a_{-1}(G) \leq n - \left\lceil \frac{\delta_n}{2} \right\rceil \quad \text{and} \quad a_0(G) \leq n - \left\lfloor \frac{\delta_n}{2} \right\rfloor .$$

Here we generalize the previous result to defensive k-alliances and we obtain lower bounds.

Theorem 1. *For every* $k \in \{-\delta_n, \dots, \delta_1\}$,

$$\left\lceil \frac{\delta_n + k + 2}{2} \right\rceil \leq a_k(G) \leq n - \left\lfloor \frac{\delta_n - k}{2} \right\rfloor .$$

Proof. Let $X \subseteq V$ be a defensive k-alliance in G. In this case, for every $v \in X$ we have

$$\delta(v) = \delta_X(v) + \delta_{\bar{X}}(v)$$
$$\delta(v) \leq \delta_X(v) + \frac{\delta(v) - k}{2}$$
$$\frac{\delta(v) + k}{2} \leq \delta_X(v) \leq |X| - 1$$
$$\frac{\delta_n + k + 2}{2} \leq |X|.$$

Hence, the lower bound follows.

On the other hand, if $X \subseteq V$ is a defensive k-alliance in G for $\delta_n \leq k \leq \delta_1$, then $a_k(G) \leq |X| \leq n \leq n - \left\lfloor \frac{\delta_n - k}{2} \right\rfloor$. Suppose $-\delta_n \leq k \leq \delta_n$. Let $S \subseteq V$ be a set of cardinality $n - \left\lfloor \frac{\delta_n - k}{2} \right\rfloor$. For every vertex $v \in S$ we have $\frac{\delta(v) - k}{2} \geq \left\lfloor \frac{\delta_n - k}{2} \right\rfloor \geq \delta_{\bar{S}}(v)$. Hence, S is a defensive k-alliance and $a_k(G) \leq |S| = n - \left\lfloor \frac{\delta_n - k}{2} \right\rfloor$. □

We denote by K_n the complete graph of order n.

Corollary 2. *For every* $k \in \{1 - n, \dots, n - 1\}$, $a_k(K_n) = \left\lceil \frac{n+k+1}{2} \right\rceil$.

Theorem 3. *For every* $k, r \in \mathbb{Z}$ *such that* $-\delta_n \leq k \leq \delta_1$ *and* $0 \leq r \leq \frac{k+\delta_n}{2}$,

$$a_{k-2r}(G) + r \leq a_k(G).$$

Proof. Let $S \subset V$ be a defensive k-alliance of minimum cardinality in G. By Theorem 1, $\frac{\delta_n+k+2}{2} \le |S|$; then we can take $X \subset S$ such that $|X| = r$. Hence, for every $v \in Y = S - X$,

$$\begin{aligned}\delta_Y(v) &= \delta_S(v) - \delta_X(v)\\ &\ge \delta_{\bar{S}}(v) + k - \delta_X(v)\\ &= \delta_{\bar{Y}}(v) + k - 2\delta_X(v)\\ &\ge \delta_{\bar{Y}}(v) + k - 2r.\end{aligned}$$

Therefore, Y is a defensive $(k-2r)$-alliance in G and, as a consequence, $a_{k-2r}(G) \le a_k(G) - r$. □

Notice that, according to the result in Corollary 2, the bound for $a_k(G)$ in Theorem 3 is attained for the complete graph K_n for every n, k, r with its respective restrictions. From the above theorem we derive some interesting consequences.

Corollary 4. *Let $t \in \mathbb{Z}$.*

- *If $\frac{1-\delta_n}{2} \le t \le \frac{\delta_1-1}{2}$, then $a_{2t-1}(G) + 1 \le a_{2t+1}(G)$.*
- *If $\frac{2-\delta_n}{2} \le t \le \frac{\delta_1}{2}$, then $a_{2(t-1)}(G) + 1 \le a_{2t}(G)$.*

Corollary 5. *For every $k \in \{0, \ldots, \delta_n\}$,*

- *if k is even, then $a_{-k}(G) + \frac{k}{2} \le a_0(G) \le a_k(G) - \frac{k}{2}$,*
- *if k is odd, then $a_{-k}(G) + \frac{k-1}{2} \le a_{-1}(G) \le a_k(G) - \frac{k+1}{2}$.*

It was shown in [5,7] that for any graph G of order n,

$$a_{-1}(G) \le \left\lceil \frac{n}{2} \right\rceil \quad \text{and} \quad a_0(G) \le \left\lfloor \frac{n}{2} \right\rfloor + 1. \tag{4}$$

By Corollary 5 and (4) we obtain the following result.

Theorem 6. *For every $k \in \{-\delta_n, \ldots, 0\}$, $a_k(G) \le \left\lceil \frac{n+k+1}{2} \right\rceil$.*

Notice that the above bound is attained, for instance, for the complete graph $G = K_n$.

3. Algebraic connectivity and defensive k-alliance number

It is well known that the second smallest Laplacian eigenvalue of a graph is probably the most important information contained in the Laplacian spectrum. This eigenvalue, frequently called the *algebraic connectivity*, is related to several important graph invariants and imposes reasonably good bounds on the values of several parameters of graphs which are very hard to compute.

The algebraic connectivity of G, μ, satisfies the following equality shown by Fiedler [4] on weighted graphs:

$$\mu = 2n \min \left\{ \frac{\sum_{v_i \sim v_j} (w_i - w_j)^2}{\sum_{v_i \in V} \sum_{v_j \in V} (w_i - w_j)^2} : w \ne \alpha \mathbf{j} \text{ for } \alpha \in \mathbb{R} \right\}, \tag{5}$$

where $V = \{v_1, v_2, \ldots, v_n\}$, $\mathbf{j} = (1, 1, \ldots, 1)$ and $w \in \mathbb{R}^n$.

The following theorem shows the relationship between the algebraic connectivity of a graph and its defensive k-alliance number.

Theorem 7. *For any connected graph G and for every $k \in \{-\delta_n, \ldots, \delta_1\}$,*

$$a_k(G) \ge \left\lceil \frac{n(\mu + k + 1)}{n + \mu} \right\rceil.$$

Proof. If S denotes a defensive k-alliance in G, then

$$\delta_{\bar{S}}(v) + k \le |S| - 1, \quad \forall v \in S. \tag{6}$$

By (5), taking $w \in \mathbb{R}^n$ defined as

$$w_i = \begin{cases} 1 & \text{if } v_i \in S;\\ 0 & \text{otherwise,} \end{cases}$$

we have

$$\mu \leq \frac{n \sum_{v \in S} \delta_{\bar{S}}(v)}{|S|(n - |S|)}. \tag{7}$$

Thus, (6) and (7) lead to

$$\mu \leq \frac{n(|S| - k - 1)}{n - |S|}. \tag{8}$$

Therefore, solving (8) for $|S|$, and considering that it is an integer, we obtain the bound on $a_k(G)$. □

The above bound is sharp as we can see in the following example. As the algebraic connectivity of the complete graph $G = K_n$ is $\mu = n$, the above theorem gives the exact value of $a_k(K_n) = \left\lceil \frac{n+k+1}{2} \right\rceil$.

Theorem 8. *For any connected graph G and for every $k \in \{-\delta_n, \dots, \delta_1\}$,*

$$a_k(G) \geq \left\lceil \frac{n\left(\mu - \left\lfloor \frac{\delta_1 - k}{2} \right\rfloor\right)}{\mu} \right\rceil.$$

Proof. If S denotes a defensive k-alliance in G, then $\delta_1 \geq \delta(v) \geq 2\delta_{\bar{S}}(v) + k$, $\forall v \in S$. Therefore,

$$\left\lfloor \frac{\delta_1 - k}{2} \right\rfloor \geq \delta_{\bar{S}}(v), \quad \forall v \in S. \tag{9}$$

Hence, by (7) and (9) the result follows. □

The bound is attained for every k in the case of the complete graph $G = K_n$.

The reader is referred to [8] for more details on the spectral study of offensive alliances and dual alliances.

4. Defensive k-alliance number and line graph

Hereafter, we denote by $\mathcal{L}(G) = (V_l, E_l)$ the line graph of a simple graph G. The degree of the vertex $e = \{u, v\} \in V_l$ is $\delta(e) = \delta(u) + \delta(v) - 2$. If the degree sequence of G is $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n$, then the maximum degree of $\mathcal{L}(G)$, denoted by Δ_l, is bounded by

$$\Delta_l \leq \delta_1 + \delta_2 - 2 \tag{10}$$

and the minimum degree of $\mathcal{L}(G)$, denoted by δ_l, is bounded by

$$\delta_l \geq \delta_n + \delta_{n-1} - 2. \tag{11}$$

In this section we obtain some results on $a_k(\mathcal{L}(G))$ in terms of the degree sequence of G.

Theorem 9. *For any simple graph G of maximum degree δ_1, and for every $k \in \{2(1 - \delta_1), \dots, 0\}$,*

$$a_k(\mathcal{L}(G)) \leq \delta_1 + \left\lceil \frac{k}{2} \right\rceil.$$

Proof. Suppose k is even. Let $v \in V(G)$ be a vertex of maximum degree in G and let $S_v = \{e \in E : v \in e\}$. Let $Y_k \subset S_v$ be such that $|Y_k| = -\frac{k}{2}$ and let $X_k = S_v - Y_k$. Thus, $\langle S_v \rangle \cong K_{\delta_1}$ and, as a consequence,

$$\delta_{X_k}(e) = \delta_1 - 1 + \frac{k}{2} \geq \delta_2 - 1 + \frac{k}{2} \geq \delta_{\bar{X}_k}(e) + k, \quad \forall e \in X_k.$$

Hence, $X_k \subset V_l$ is a defensive k-alliance in $\mathcal{L}(G)$. So, for k even we have $a_k(\mathcal{L}(G)) \leq \delta_1 + \left\lceil \frac{k}{2} \right\rceil$. Moreover, if k is odd, then $a_k(\mathcal{L}(G)) \leq a_{k+1}(\mathcal{L}(G)) \leq \delta_1 + \left\lceil \frac{k+1}{2} \right\rceil = \delta_1 + \left\lceil \frac{k}{2} \right\rceil$. □

One advantage of applying the above bound is that it requires only little information about the graph G; just the maximum degree. The above bound is tight, as we will see below, for any δ-regular graph, and $k \in \{2(1-\delta), \dots, 0\}$, $a_k(\mathcal{L}(G)) = \delta + \left\lceil \frac{k}{2} \right\rceil$. Even so, we can improve this bound for the case of nonregular graphs. The drawback of such improvement is that we need to know more about G.

Theorem 10. *Let G be a simple graph, whose degree sequence is $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n$. Let $v \in V$ be such that $\delta(v) = \delta_1$, let $\delta_v = \max_{u:u\sim v}\{\delta(u)\}$ and let $\delta_* = \min_{v:\delta(v)=\delta_1}\{\delta_v\}$. For every $k \in \{2-\delta_*-\delta_1, \ldots, \delta_1-\delta_*\}$,*

$$a_k(\mathcal{L}(G)) \leq \left\lceil \frac{\delta_1+\delta_*+k}{2} \right\rceil.$$

Moreover, for every $k \in \{2-\delta_1-\delta_2, \ldots, \delta_1+\delta_2-2\}$,

$$\left\lceil \frac{\delta_n+\delta_{n-1}+k}{2} \right\rceil \leq a_k(\mathcal{L}(G)).$$

Proof. Let $v \in V$ be a vertex of maximum degree $\delta(v) = \delta_1$ such that v is adjacent to a vertex of degree δ_*. Let $S_v = \{e \in E : v \in e\}$. Suppose $\delta_1+\delta_*+k$ is even. Therefore, taking $S \subset S_v$ such that $|S| = \frac{\delta_1+\delta_*+k}{2}$, we obtain $\langle S\rangle \cong K_{\frac{\delta_1+\delta_*+k}{2}}$. Thus, $\forall e \in S$,

$$\delta_S(e) - k = \frac{\delta_1+\delta_*+k}{2} - 1 - k \geq \delta_1+\delta_*-1-\frac{\delta_1+\delta_*+k}{2}.$$

On the other hand, as $\delta(e) = \delta_S(e) + \delta_{\bar{S}}(e)$, we have

$$\delta_1+\delta_*-2 \geq \frac{\delta_1+\delta_*+k}{2} - 1 + \delta_{\bar{S}}(e) \Leftrightarrow \delta_1+\delta_*-1-\frac{\delta_1+\delta_*+k}{2} \geq \delta_{\bar{S}}(e).$$

So, $\delta_S(e) - k \geq \delta_{\bar{S}}(e)$ and S is a defensive k-alliance in $\mathcal{L}(G)$ and, as a consequence, $a_k(\mathcal{L}(G)) \leq \left\lceil \frac{\delta_1+\delta_*+k}{2} \right\rceil$. If $\delta_1+\delta_*+k$ is odd, then $\delta_1-\delta_* > k$. Therefore, $a_k(\mathcal{L}(G)) \leq a_{k+1}(\mathcal{L}(G)) \leq \left\lceil \frac{\delta_1+\delta_*+k+1}{2} \right\rceil = \left\lceil \frac{\delta_1+\delta_*+k}{2} \right\rceil$.

The lower bound follows from Theorem 1 and (11). □

Corollary 11. *For any δ-regular graph, $\delta > 0$, and for every $k \in \{2(1-\delta), \ldots, 0\}$,*

$$a_k(\mathcal{L}(G)) = \delta + \left\lceil \frac{k}{2} \right\rceil.$$

We recall that a graph $G = (V, E)$ is a (δ_1, δ_2)-semiregular bipartite graph if the set V can be partitioned into two disjoint subsets V_1, V_2 such that if $u \sim v$ then $u \in V_1$ and $v \in V_2$ and also $\delta(v) = \delta_1$ for every $v \in V_1$ and $\delta(v) = \delta_2$ for every $v \in V_2$.

Corollary 12. *For any (δ_1, δ_2)-semiregular bipartite graph G, $\delta_1 > \delta_2$, and for every $k \in \{2-\delta_1-\delta_2, \ldots, \delta_1-\delta_2\}$,*

$$a_k(\mathcal{L}(G)) = \left\lceil \frac{\delta_1+\delta_2+k}{2} \right\rceil.$$

We should point out that from the results obtained in the other sections of this work on $a_k(G)$, we can derive some new results on $a_k(\mathcal{L}(G))$. The reader is referred to [11] for more details on $a_{-1}(\mathcal{L}(G))$ and $a_0(\mathcal{L}(G))$.

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