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RESEARCH ARTICLE

Towards a better understanding of the semigroup tree

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Abstract In this paper we elaborate on the structure of the semigroup tree and the regularities on the number of descendants of each node observed earlier by the first author. These regularities admit two different types of behavior and in this work we investigate which of the two types takes place for some well-known classes of semigroups. Also we study the question of what kind of chains appear in the tree and characterize the properties (like being (in)finite) thereof. We conclude with some thoughts that show how this study of the semigroup tree may help in solving the conjecture of Fibonacci-like behavior of the number of semigroups with given genus.

Keywords Numerical semigroup · Fibonacci numbers

1 Introduction

A numerical semigroup is a subset of the non-negative integers $\mathbb{N}_0$ which is closed under addition, contains 0 and its complement in $\mathbb{N}_0$ is finite. The elements in this

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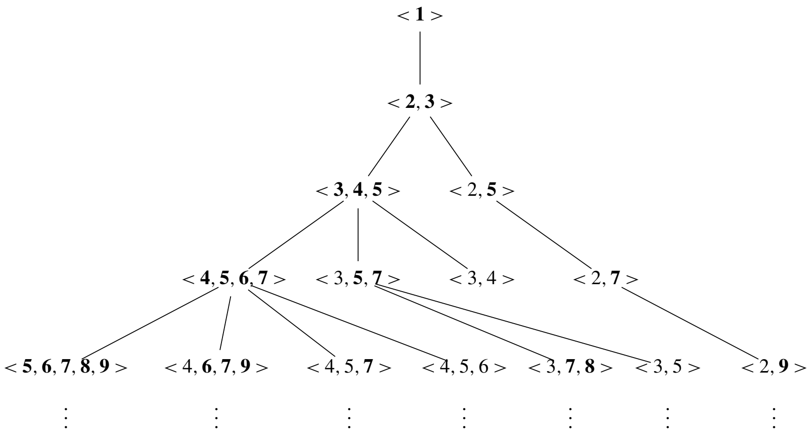

Fig. 1 Recursive construction of the numerical semigroups of genus g from the numerical semigroups of genus $g-1$. Generators larger than the conductor are written in bold face

complement are called *gaps* and the number of gaps of a numerical semigroup is its *genus*. The smallest integer in a numerical semigroup from which all larger integers belong to the numerical semigroup is called the *conductor* of the numerical semigroup. Notice that if a numerical semigroup is different from $\mathbb{N}_0$ then the conductor of this numerical semigroup is exactly the largest gap (known as its *Frobenius number*) plus one. A good reference for numerical semigroups and related results is [14].

It can be shown that each numerical semigroup has a unique minimal set of generators. From now on we write "generator" for an element of this minimal set. The numerical semigroups of genus g can be obtained from the numerical semigroups of genus $g-1$ by taking out one by one the generators that are larger than or equal to the conductor of each semigroup. This leads to an infinite tree containing all numerical semigroups, with root corresponding to the trivial semigroup and where each level of nodes represents numerical semigroups of genus given by the level. The parent of a numerical semigroup is obtained by adding to the semigroup its Frobenius number. This tree is illustrated in Fig. 1, where we used $\langle a_1, \dots, a_k\rangle$ to denote the numerical semigroup generated by $a_1, \dots, a_k$. This construction was already considered in [12, 17, 18].

The number n_g of all numerical semigroups of genus g has been studied in [3–5, 8, 11]. In [4] it is conjectured that n_g asymptotically behaves like the Fibonacci numbers. That is, $n_g \geq n_{g-1} + n_{g-2}$, $\lim_{g\to\infty}(n_{g-1}+n_{g-2})/n_g = 1$, and n_g/n_{g-1} approaches the golden ratio. In [5] the tree of numerical semigroups is used to derive, for $g \geq 3$, the bounds $2F_g \leq n_g \leq 1 + 3\cdot 2^{g-3}$, where F_g denotes the g-th Fibonacci number. The goal of this paper is providing results for better understanding the semigroup tree and giving possible directions for attacking the previous conjecture. The bounds given in [5] are a consequence of the fact that only two kinds of generators exist in a numerical semigroup larger than or equal to its conductor. In Sect. 2 we call these two kinds of generators *weak* and *strong* and we study their existence in three well-known classes of numerical semigroups: symmetric, pseudo-symmetric, and Arf semigroups.

In Sect. 3 we analyze which nodes have an infinite number of descendants. For the nodes having a finite number of descendants we give a way to determine the descendant at largest distance; for the nodes having an infinite number of descendants we determine the number of infinite chains in which the semigroup lies. It turns out here that primality and coprimality of integers appear in the scene as discriminating factors. Some results related to weak and strong generators of semigroups lying in infinite chains are also given.

In the last section we give what we think should be future directions for attacking the conjecture on the Fibonacci-like behavior of n_g and how the results presented in the first sections could help.

2 Behavior of some known classes of numerical semigroups

The enumeration λ of a numerical semigroup Λ is the unique increasing bijective map $\mathbb{N}_0 \to \Lambda$. Usually $\lambda(i)$ is denoted λ_i. It is easy to check that if c and g are the conductor and the genus of Λ then $\lambda_{c-g} = c$ and for $\lambda_i \geq c$, $\lambda_i = i + g$. A semigroup for which $\lambda_1 = c$, i.e. a semigroup of the form $\{0\} \cup [c, \infty)$, is called *ordinary*.

It was shown in [5] that the next lemma holds.

Lemma 1 *If $\lambda_{i_1} < \lambda_{i_2} < \cdots < \lambda_{i_n}$ are the generators of a non-ordinary numerical semigroup Λ that are larger than or equal to its conductor then the generators of $\Lambda \setminus \{\lambda_{i_j}\}$ that are larger than or equal to its conductor are either $\lambda_{i_{j+1}}, \ldots, \lambda_{i_n}$ or $\lambda_{i_{j+1}}, \ldots, \lambda_{i_n}, \lambda_{i_j} + \lambda_1$.*

Motivated by this lemma, we call the generators of a non-ordinary numerical semigroup that are larger than or equal to its conductor, the *effective generators* and we say that an effective generator λ_{i_j} is *strong* if the set of effective generators of $\Lambda \setminus \{\lambda_{i_j}\}$ is $\lambda_{i_{j+1}}, \ldots, \lambda_{i_n}, \lambda_{i_j} + \lambda_1$. An effective generator that is not strong is called a *weak generator*.

Finally we say that a *leaf* is a node with no descendants, a *stick* is a node with exactly one descendant and a *bush* is a node with two or more descendants.

2.1 Symmetric semigroups

Symmetric semigroups are those numerical semigroups for which the conductor is twice the genus. Symmetric semigroups and their applications to coding theory have been studied, among others, in [2, 6, 9, 10]. An important property of symmetric semigroups is that if c is the conductor of a symmetric semigroup Λ then any integer i is a gap of Λ if and only if $c - 1 - i$ is a non-gap.

The semigroups of the form $\langle 2, 2n + 1\rangle, n \geq 1$ are symmetric. They are called *hyperelliptic semigroups*.

Lemma 2 *Hyperelliptic numerical semigroups are sticks and the unique effective generator, which is the conductor plus one, is strong.*

Apéry introduced in [1] what later was called the Apéry set of a numerical semigroup. Given a numerical semigroup Λ and a non-negative integer m, the Apéry set of Λ associated to m is the set $\omega_0^{(m)} = \{0, \omega_1^{(m)}, \dots, \omega_{m-1}^{(m)}\}$ of the smallest elements of Λ belonging to the different congruence classes mod m, given in increasing order. An equivalent definition is the set of non-gaps which are a gap plus m. If m is the smallest non-zero non-gap, then the union of the Apéry set associated to m with $\{m\}$ contains the generators of the semigroup. Furthermore, the Frobenius number of a numerical semigroup is $\omega_{m-1}^{(m)} - m$ for all m.

He proved in the same reference that a numerical semigroup is symmetric if and only if its Apéry sets satisfy that if $i + j = m - 1$ then $\omega_i^{(m)} + \omega_j^{(m)} = \omega_{m-1}^{(m)}$. We will use this fact for proving the next lemma.

Lemma 3 *Non-hyperelliptic symmetric semigroups are leaves.*

Proof Suppose that m is the first non-zero non-gap and thus the Apéry set of Λ associated to m contains all the generators of Λ except m. Suppose that Λ has a descendant. Then there is a generator and so an element of the Apéry set larger than the Frobenius number. That is, there exists i such that $\omega_{m-1}^{(m)} - m < \omega_i^{(m)} \leq \omega_{m-1}^{(m)}$. By the property of Apéry sets of symmetric semigroups, $\omega_{m-1}^{(m)} - \omega_i^{(m)}$ also belongs to the Apéry set, but by the inequality $\omega_{m-1}^{(m)} - m < \omega_i^{(m)}$, we have $\omega_{m-1}^{(m)} - \omega_i^{(m)} < m$ and so $\omega_{m-1}^{(m)} = \omega_i^{(m)}$. Consequently, only $\omega_{m-1}^{(m)}$ can be an effective generator. But, again by the property of Apéry sets of symmetric semigroups, $\omega_{m-1}^{(m)}$ can only be a generator if $m \leq 2$. $\square$

As an example of non-hyperelliptic symmetric semigroup consider $\Lambda = \{0, 4, 5, 8, 9, 10\} \cup [12, \infty)$. In this case the generators are 4 and 5 and none of them is effective.

2.2 Pseudo-symmetric semigroups

Pseudo-symmetric semigroups are those numerical semigroups for which the conductor is twice the genus minus one [13]. Notice that as opposite to symmetric semigroups, in this case the conductor needs to be an odd number. An important property of pseudo-symmetric semigroups analogous to the one for symmetric semigroups is that if c is the conductor of a pseudo-symmetric semigroup Λ then any integer i different from $(c-1)/2$ is a gap of Λ if and only if $c - 1 - i$ is a non-gap. Furthermore, for pseudo-symmetric semigroups the Apéry sets satisfy a property analogous to that of symmetric semigroups. First of all notice that given a numerical semigroup, half of its Frobenius number is a gap, and so $\frac{c-1}{2} + m$ belongs to its Apéry set associated to m. It is proved in [13, Proposition 5] that if $\bar{\omega}_0^{(m)} = 0, \bar{\omega}_1^{(m)}, \dots, \bar{\omega}_{m-2}^{(m)}$ is the Apéry set without $\frac{c-1}{2} + m$, then $\bar{\omega}_i^{(m)} + \bar{\omega}_j^{(m)} = \bar{\omega}_{m-2}^{(m)}$ whenever $i + j = m - 2$.

Lemma 4

(1) *The unique pseudo-symmetric semigroup of genus g with only one interval of non-gaps between* 0 *and the conductor is* $\Lambda_{ps_g} = \{0, g, g+1, \ldots, 2g-3\} \cup [2g-1, \infty)$.
(2) *The numerical semigroup* $\Lambda_{ps_3} = \{0, 3\} \cup [5, \infty)$, *has* 5 *and* 7 *as the only effective generators. The generator* 5 *is strong and the generator* 7 *is weak.*
(3) *The numerical semigroup* $\Lambda_{ps_4} = \{0, 4, 5\} \cup [7, \infty)$, *has* 7 *as the only effective generator and it is strong.*
(4) *The numerical semigroup* Λ_{ps_g}, *for* $g \geq 5$ *is a stick, its unique effective generator is* c, *and it is weak.*

Proof The proof of statement (1) follows directly from the fact that if c is the conductor of a pseudo-symmetric semigroup Λ then any integer i different from $(c-1)/2$ is a gap of Λ if and only if $c-1-i$ is a non-gap. Statements (2) and (3) can be proved by an exhaustive search of generators and by checking which are weak and which are strong.

Since the conductor of Λ_{ps_g} is $2g-1$, every integer larger than or equal to $4g-2$ will not be a generator. The integer $4g-3$ is not a generator since $4g-3 = g + (2g-3)$. The integer $4g-4$ is not a generator since $4g-4 = (2g-3) + (2g-1)$. The integers from $2g$ to $4g-6$ are generated by the interval $g, \ldots, 2g-3$. So the only effective generator of Λ_{ps_g} can be $c = 2g-1$ and $4g-5$. It is easy to check that c is a generator. If the integer $4g-5$ is larger than or equal to $g + (2g-1)$ then it is not a generator. This is equivalent to $g \geq 4$.

On the other hand, $2g-1$ is weak if and only if $g + (2g-1)$ is a sum of two non-gaps strictly smaller than $2g-1$ and this is equivalent to having $g + g \leq g + (2g-1) \leq (2g-3) + (2g-3)$, which in turn is equivalent to $g \geq 5$. Thus c is a weak generator if $g \geq 5$. □

Lemma 5

(1) *A numerical semigroup is pseudo-symmetric and has* $\lambda_1 = 3$ *if and only if it is equal to* $\Lambda = \{0, 3, 6, \ldots, 3k, 3(k+1)-1, 3(k+1), 3(k+2)-1, 3(k+2), \ldots, 3(2k-1)-1, 3(2k-1)\} \cup [3(2k-1)+2, \infty)$ *or* $\Lambda = \{0, 3, 6, \ldots, 3k, 3(k+1), 3(k+1)+1, 3(k+2), 3(k+2)+1, \ldots, 3(2k), 3(2k)+1\} \cup [3(2k)+3, \infty)$ *for some* k.
(2) *Each pseudo-symmetric semigroup with* $\lambda_1 = 3$ *has a unique effective generator, it is* $c+2$ *and it is weak.*
(3) *The descendants of a pseudo-symmetric semigroups with* $\lambda_1 = 3$ *are non-hyperelliptic symmetric semigroups, and thus, leaves.*

Proof

(1) From the property of pseudo-symmetric semigroups that any non-negative integer i different from $(c-1)/2$ is a gap if and only if $c-1-i$ is a non-gap we deduce that each pseudo-symmetric semigroup with $\lambda_1 = 3$ must be one of the semigroups above. To see that these semigroups are always pseudo-symmetric, let us compute the genus and the conductor. In the first case we have that up to $3k$ the semigroup Λ has exactly $2k$ gaps: 2 gaps per interval

$[3i, 3i+2]$, $0 \le i \le k-1$. Then from $3k+1$ to $3(2k-1)$ there are $k-1$ gaps: one per interval $[3i+1, 3(i+1)]$, $k \le i \le 2k-2$. Together with the gap $3(2k-1)+1$ that makes $g = 2k+k-1+1 = 3k$ gaps. Obviously $c = 3(2k-1)+2 = 2g-1$. So Λ is pseudo-symmetric. The other case is done analogously and we have $g = 2(k+1)+k = 3k+2$, $c = 3(2k)+3 = 2g-1$.

(2) Since $3 \in \Lambda$, the only elements larger than or equal to the conductor that can be generators are $c, c+1, c+2$. The elements c and $c+1$ cannot be generators, because $c = 3(2k-1)+2 = 3(2k-1)-1+3$, $c+1 = 3(2k-1)+3$ for the first case and similarly is done for the second. Let us show that $c+2$ is a generator. Consider the first case, the second one is done analogously. We have $c+2 = 3(2k-1)+4 = 6k+1$, so it has residue 1 modulo 3. Note that all the non-gaps less than $c+2$ have residues 0 or 2 modulo 3. So, if $c+2$ is not a generator, it is a sum of two non-gaps with residue 2. So we have $c+2 = 3(k+i)-1+3(k+j)-1 = 6k+3i+3j-2$ for some $i, j \ge 1$. But then we have that $i+j=1$, a contradiction.

To see that $c+2$ is a weak generator, let us analyze $(c+2)+3$ in both cases. In the first case it is equal to $3(2k-1)+2+2+3 = 6k+4 = 2(3k+2)$, which is not a generator. In the second case it is equal to $3(2k)+3+2+3 = 6k+8 = 2(3(k+1)+1)$, which is not a generator either.

(3) The only descendant of Λ is obtained by removing $c+2$. The semigroup $\Lambda \setminus \{c+2\}$ is symmetric since its genus is $g+1$ and its conductor is $c+3$, and we have $c+3 = 2(g+1)$, since $c = 2g-1$. Obviously, $\Lambda \setminus \{c+2\}$ is a non-hyperelliptic semigroup, and thus a leaf, cf. Lemma 3. □

Lemma 6 *Each pseudo-symmetric semigroup with* $\lambda_1 \ne 3$ *and with more than one interval of non-gaps between* 0 *and the conductor is a leaf.*

Proof Suppose that m is the first non-zero non-gap and thus the union of the Apéry set of Λ associated to m with $\{m\}$ contains all the generators of Λ. On one hand, since there are at least two intervals of non-gaps between 0 and c, and by pseudo-symmetry properties, we have $m < \frac{c-1}{2}$, and so $\frac{c-1}{2} + m < c-1$. Suppose now that Λ has a descendant. Then there is a generator and so an element of the Apéry set different from $\frac{c-1}{2}+m$ and larger than the Frobenius number. Let us denote the elements of the Apéry set which are different from $\frac{c-1}{2}+m$ by $0 = \bar{\omega}_0^{(m)} < \bar{\omega}_1^{(m)} < \cdots < \bar{\omega}_{m-2}^{(m)}$. There exists i such that $\bar{\omega}_{m-2}^{(m)} - m < \bar{\omega}_i^{(m)} \le \bar{\omega}_{m-2}^{(m)}$. By the property of Apéry sets of pseudo-symmetric semigroups, $\bar{\omega}_{m-2}^{(m)} - \bar{\omega}_i^{(m)}$ also belongs to the Apéry set, but by the inequality $\bar{\omega}_{m-2}^{(m)} - m < \bar{\omega}_i^{(m)}$, we have $\bar{\omega}_{m-2}^{(m)} - \bar{\omega}_i^{(m)} < m$ and so $\bar{\omega}_{m-2}^{(m)} = \bar{\omega}_i^{(m)}$. Consequently, only $\bar{\omega}_{m-2}^{(m)}$ can be an effective generator. But, again by the property of Apéry sets of pseudo-symmetric semigroups, $\bar{\omega}_{m-2}^{(m)}$ can only be a generator if $m \le 3$. □

As an example of pseudo-symmetric semigroup with $\lambda_1 \ne 3$ and with more than one interval of non-gaps between 0 and the conductor we can take $\Lambda = \{0, 4, 7, 8, 9\} \cup [11, \infty)$. In this case the generators are $4, 7, 9$ and none of them is effective.

A numerical semigroup is said to be *irreducible* if it cannot be expressed as an intersection of two numerical semigroups properly containing it. It was proven in [13] that irreducible semigroups are exactly symmetric and pseudo-symmetric semigroups. Thus we have shown that the only non-leaves corresponding to irreducible numerical semigroups are those treated in Lemmas 2, 4, 5. Moreover the number of effective generators is small and the number of strong generators is even smaller. Therefore, the parts of the semigroup tree in a vicinity of an irreducible semigroup are not "bushy" and are easily described.

2.3 Arf semigroups

A numerical semigroup Λ with enumeration λ is said to be Arf if $\lambda_i + \lambda_j - \lambda_k \in \Lambda$ for every $i, j, k \in \mathbb{N}_0$ with $i \geq j \geq k$. Hyperelliptic semigroups are examples of Arf semigroups. In fact, it was shown in [7] that hyperelliptic semigroups are the only Arf symmetric semigroups. A lot of work has been done related to Arf semigroups. One can see, for instance, [2, 7, 15, 16].

For the next lemma we use the fact that for an Arf numerical semigroup Λ, an element λ_i different from 0 and different from λ_1 is a generator if and only if $\lambda_i - \lambda_1 \notin \Lambda$.

Lemma 7

(1) *Non-hyperelliptic Arf numerical semigroups are bushes.*
(2) *Arf semigroups appear as descendants of semigroups with strong generators when removing one such generator.*

Proof

(1) For an Arf semigroup we know that if $i, i+1 \in \Lambda$, then $i \geq c$. Indeed, for $j \geq i$,

$$\begin{aligned} j &= i + \overbrace{((i+1)-i) + ((i+1)-i) + \cdots + ((i+1)-i)}^{(j-i)} \\ &= \underbrace{\underbrace{\underbrace{i + ((i+1)-i)}_{\in\Lambda} + ((i+1)-i)}_{\in\Lambda} + \cdots + ((i+1)-i)}_{\in\Lambda}. \end{aligned}$$

Thus we know that $c-1$ and either $c-2$ or $c-3$ are gaps. Since Λ is not hyperelliptic, $\lambda_1 \geq 3$. Thus, $c-1+\lambda_1$ and either $c-2+\lambda_1$ or $c-3+\lambda_1$ are generators.

(2) It follows from the remark previous to the Lemma. □

It was shown in [15] that at most two of the descendants of Arf semigroups are Arf. For illustrating this, notice that $\{0,5,7\} \cup [9,\infty)$ has no Arf descendants; $\{0,5\} \cup [7,\infty)$ has two Arf descendants: $\{0,5\} \cup [8,\infty)$ and $\{0,5,7\} \cup [9,\infty)$; $\{0,5\} \cup [10,\infty)$ has one Arf descendant: $\{0,5,10\} \cup [12,\infty)$.

3 Infinite chains

We say that an infinite sequence of numerical semigroups $\Lambda_0 = \mathbb{N}_0, \Lambda_1, \Lambda_2, \dots$ is an *infinite chain* if for each $i \geq 1$, Λ_{i-1} can be obtained by adding to Λ_i its Frobenius number. Clearly, a numerical semigroup has infinitely many descendants in the semigroup tree if and only if it lies in an infinite chain.

Let $\mathbb{S}$ be the set of all numerical semigroups, and let $\mathbb{I}$ be the set of all infinite chains. One element of $\mathbb{I}$ contains infinitely many elements in $\mathbb{S}$. Each element in $\mathbb{S}$ may be contained in none, one, several or infinitely many elements of $\mathbb{I}$. The aim of this section is analyzing what circumstances give what of these cases. Then, we analyze what kind of generators appear in infinite chains.

For the proof of the next lemma we will use that the integers $l_1, \dots, l_m$ generate a numerical semigroup if and only if they are coprime.

Lemma 8 *Given an infinite chain* $(\Lambda_i)_{i \geq 0}$,

$$\bigcap_{i \geq 0} \Lambda_i = d \cdot \Lambda$$

for some integer $d > 1$ *and some numerical semigroup* Λ.

Proof The intersection $\bigcap_{i \geq 0} \Lambda_i$ satisfies $0 \in \bigcap_{i \geq 0} \Lambda_i$ and $x + y \in \bigcap_{i \geq 0} \Lambda_i$ for all $x, y \in \bigcap_{i \geq 0} \Lambda_i$. Furthermore, all elements in $\bigcap_{i \geq 0} \Lambda_i$ must be divisible by an integer $d > 1$. Indeed, otherwise we could find a finite set of coprime elements which would generate a numerical semigroup, and this numerical semigroup should be a subset of $\bigcap_{i \geq 0} \Lambda_i$. Then the infinite chain would not contain any semigroup with genus larger than that of this semigroup, giving a contradiction. Let d be the greatest of the common divisors of $\bigcap_{i \geq 0} \Lambda_i$. Then $\frac{1}{d}(\bigcap_{i \geq 0} \Lambda_i)$ must be a numerical semigroup. □

Lemma 9 *Given an integer* $d > 1$ *and a numerical semigroup* Λ *the infinite chain obtained by deleting repetitions in the sequence* $\Lambda_j = d \cdot \Lambda \cup \{l \in \mathbb{N} : l \geq j\}$ *has intersection* $d \cdot \Lambda$.

Lemma 8 and its proof suggest the map

$$\begin{array}{rcl} \sigma : \mathbb{I} & \longrightarrow & \mathbb{N}_{\geq 2} \times \mathbb{S} \\ \{\Lambda_i\}_{i \in \mathbb{N}_0} & \mapsto & \left(\gcd\left(\bigcap_{i \geq 0} \Lambda_i \right), \bigcap_{i \geq 0} \Lambda_i / \gcd\left(\bigcap_{i \geq 0} \Lambda_i \right) \right). \end{array}$$

Lemma 9 proves that the map

$$\begin{array}{rcl} \mathbb{N}_{\geq 2} \times \mathbb{S} & \longrightarrow & \mathbb{I} \\ (d, \Lambda) & \mapsto & \{d \cdot \Lambda \cup \{l \in \mathbb{N} : l \geq i\}\}_{i \notin d \cdot \Lambda} \end{array}$$

is the inverse of σ.

Consequently, $\mathbb{I}$ and $\mathbb{N}_{\geq 2} \times \mathbb{S}$ are in a one-to-one correspondence.

In the next theorem we show that the greatest common divisor of the first elements of a numerical semigroup determine whether the numerical semigroup has infinite number of descendants. Notice that since $\lambda_{c-g} = c$, the set $\lambda_0, \dots, \lambda_{c-g-1}$ is the set of non-gaps smaller than the conductor.

Theorem 10 *Let Λ be a numerical semigroup with enumeration λ, genus g, and conductor c, and let d be the greatest common divisor of $\lambda_0, \dots, \lambda_{c-g-1}$. Then,*

(1) *Λ lies in an infinite chain if and only if $d \neq 1$.*
(2) *If $d = 1$ then the descendant of Λ with largest genus is the numerical semigroup generated by $\lambda_0, \dots, \lambda_{c-g-1}$.*
(3) *If $d \neq 1$ then Λ lies in infinitely many infinite chains if and only if d is not prime.*
(4) *If d is a prime then the number of infinite chains in which Λ lies is the number of descendants of $\{\frac{\lambda_0}{d}, \frac{\lambda_1}{d}, \dots, \frac{\lambda_{c-g-1}}{d}\} \cup \{l \in \mathbb{N}_0 : l \geq \lceil \frac{c}{d} \rceil\}$.*

Proof

(1) If $d = 1$ then $\lambda_0, \dots, \lambda_{c-g-1}$ generate a numerical semigroup Λ' and each descendant of Λ must contain Λ'. Thus, the maximum of the genus of the descendants is the genus of Λ' which is finite. On the other hand, if $d \neq 1$ then

$$\lambda_0 = d\tilde{\lambda}_0, \dots, \lambda_{c-g-1} = d\tilde{\lambda}_{c-g-1}$$

with $\tilde{\lambda}_0, \dots, \tilde{\lambda}_{c-g-1}$ coprime. Let $\tilde{\Lambda}$ be the numerical semigroup generated by $\tilde{\lambda}_0, \dots, \tilde{\lambda}_{c-g-1}$. Consider the sequence of semigroups

$$\Lambda_i = d \cdot \tilde{\Lambda} \cup \{l \in \mathbb{N}_0 : l \geq i\}.$$

By deleting repetitions we obtain an infinite chain that contains Λ.
(2) It follows from the proof of the statement (1) above.
(3) If d is not prime then $d = d_1 d_2$ for some $d_1, d_2 > 1$ and, as before,

$$\lambda_0 = d_1 d_2 \tilde{\lambda}_0, \dots, \lambda_{c-g-1} = d_1 d_2 \tilde{\lambda}_{c-g-1}$$

with $\tilde{\lambda}_0, \dots, \tilde{\lambda}_{c-g-1}$ coprime. Let $\tilde{\Lambda}$ be the numerical semigroup generated by $\tilde{\lambda}_0, \dots, \tilde{\lambda}_{c-g-1}$. For each $i \geq 0$ and each $j \geq 0$ define

$$\Lambda_{i,j} = d_1 d_2 \tilde{\Lambda} \cup \{d_1 l \in \mathbb{N}_0 : l \geq i\} \cup \{l \in \mathbb{N}_0 : l \geq j\}.$$

For each fixed $i \geq \lceil \frac{c}{d_1} \rceil$, by deleting repetitions in the sequence $\left(\Lambda_{i,j}\right)_{j \geq 0}$ we obtain an infinite chain. Moreover every such chain contains Λ, as $\Lambda = \Lambda_{i,c}$, if $i \geq \lceil \frac{c}{d_1} \rceil$. For each $i \geq \lceil \frac{c}{d_1} \rceil$ this chain is different. Thus we get infinitely many infinite chains. The complete result in this statement follows from statement 4.
(4) Suppose that an infinite chain $(\Lambda_i)_{i \geq 0}$ contains Λ. It must satisfy $\bigcap_{i \geq 0} \Lambda_i = d \cdot \tilde{\Lambda}$ for a unique numerical semigroup $\tilde{\Lambda}$ such that

- $d\tilde{\lambda}_0 = \lambda_0, \dots d\tilde{\lambda}_{c-g-1} = \lambda_{c-g-1},$

- $d\tilde{\lambda}_{c-g} \geq c$, since $d\tilde{\Lambda} \subseteq \Lambda$.

Thus, $\tilde{\Lambda}$ is a descendant of $\{\frac{\lambda_0}{d}, \frac{\lambda_1}{d}, \ldots, \frac{\lambda_{c-g-1}}{d}\} \cup \{l \in \mathbb{N}_0 : l \geq \lceil \frac{c}{d} \rceil\}$. □

Lemma 11 *Let Λ be a numerical semigroup with enumeration λ, genus g, conductor c, and $\gcd(\lambda_0, \ldots, \lambda_{c-g-1}) = d > 1$ lying in an infinite chain. Then*

(1) *All non-gaps between c and $c + \lambda_1 - 1$ that are not multiples of d are generators. Thus Λ has at least $\lambda_1 - \frac{\lambda_1}{d}$ effective generators.*
(2) *If there are at least two non-gaps between 0 and c, then all non-gaps between c and $c + d - 1$ that are not multiples of d are strong generators. Thus Λ has at least $d - 1$ strong generators.*
(3) *If there is just one non-gap between 0 and c, then there is at least one strong generator.*

Proof

(1) If $c \leq \lambda_k \leq c + \lambda_1 - 1$, λ_k is not a multiple of d, and there exist $0 < i < j$ such that $\lambda_i + \lambda_j = \lambda_k$ then it must be $\lambda_j < c$; otherwise $\lambda_k = \lambda_i + \lambda_j \geq \lambda_1 + c$. But if $\lambda_i, \lambda_j < c$ then $\lambda_k = \lambda_i + \lambda_j$ is a multiple of d, since λ_i and λ_j are, a contradiction.
(2) If $c \leq \lambda_k \leq c + d - 1$, λ_k is not a multiple of d, and there exist $1 < i < j$ such that $\lambda_i + \lambda_j = \lambda_1 + \lambda_k$ then it must be $\lambda_i < c$. Otherwise $\lambda_2 + c > \lambda_1 + c + d - 1 \geq \lambda_1 + \lambda_k = \lambda_i + \lambda_j \geq 2c$, a contradiction since $\lambda_2 \leq c$. But then λ_i is a multiple of d and $\lambda_i + \lambda_j = \lambda_1 + \lambda_k$ means that $\lambda_j \equiv \lambda_k \bmod d$. By hypothesis λ_k is not a multiple of d and so λ_j is not a multiple of d either and consequently $\lambda_j \geq c$. But then $\lambda_k - \lambda_j \leq c + d - 1 - c = d - 1$, so $\lambda_j = \lambda_k$ and $\lambda_i = \lambda_1 + \lambda_k - \lambda_j = \lambda_1$, a contradiction.
(3) For the last statement notice that at least c or $c + 1$ is strong. □

Notice that in the second statement of the previous lemma the requirement that there are at least two non-gaps between 0 and c is necessary. As a counterexample consider the semigroup $\{0, 8\} \cup [10, \infty)$. In this case, $d = \lambda_1 = 8$ and all non-gaps between 10 and $10 + \lambda_1 - 1 = 17$ are generators except for 16 which is a multiple of d. This is a consequence of the first statement. The second statement fails since 12 is between c and $c + d - 1$ and it is not a multiple of d, but $12 + 8 = 10 + 10$.

4 Future directions for solving the conjecture about the Fibonacci-like behavior of n_g

In this section we outline some further thoughts on strong/weak generators and how they might help to solve the Fibonacci-like problem. First of all, computational evidence suggests that as g grows, the portion of strong generators among all effective generators becomes smaller. Namely, the following is conjectured.

Conjecture 12 *Let S_g be the number of all strong generators in all numerical semigroups of genus g and let W_g be the number of all weak generators in all numerical*

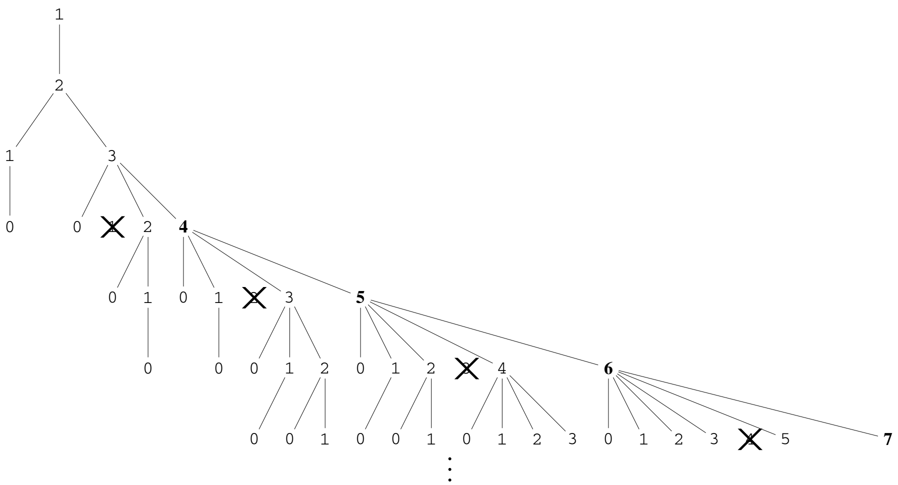

Fig. 2 Tree A. It is a subtree of the tree of numerical semigroups

semigroups of genus g. We conjecture that

$$\lim_{g\to\infty}\frac{S_g}{W_g}=0.$$

Notice that by Lemma 1, if the number of effective generators (and so the number of descendants) of a semigroup is k and all k effective generators are weak then the number of effective generators (and so the number of descendants) of its descendants is respectively $0, 1, \dots, k-1$. In [5] the tree A represented in Fig. 2 was recursively defined as follows: Its root is labeled as 1 and it has a single descendant which is labeled as 2. This descendant in turn has two descendants labeled as 1 and 3. At each level g, the number of descendants of a node is equal to its label. From level $g=2$ on, if the label of a node is k then the labels of its descendants are $0,\dots,k-1$ except for the node with label $k=g+1$, whose descendants have labels $0,\dots,k-3,k-1,k+1$.

Because of Lemma 1 and because of the particular structure of ordinary semigroups, the semigroup tree in Fig. 1 contains A as a subtree.

Define $A_0=\{1\}$, $A_1=\{2\}$ and for $g\geq 2$ define A_g as

$$A_g=\{g+1\}\cup\left(\bigcup_{m\in A_{g-1}}\{0,1,\dots,m-1\}\right)\setminus\{g-2\}.$$

The tree A has A_g as the nodes at distance g from its root. Thus, $|A_g|\leq n_g$. It was shown in [5] that $|A_g|=2F_g$, where F_i denotes the i-th Fibonacci number. From this the lower bound $n_g\geq 2F_g$ was deduced.

The next Proposition observes that no matter how a tree behaves at the beginning, if at some point its generation rule coincides with the one of A, the Fibonacci behavior is observed from some point on.

Proposition 13 *Let $l \geq 2$ be an integer and let L_l be a multiset composed of some (maybe with repetitions) numbers $\leq l-2$, and numbers $l-1$ and $l+1$. For $k > l$ define recursively*

$$L_k = \{k+1\} \bigcup \left(\bigcup_{m \in L_{k-1}} \{0, 1, \ldots, m-1\} \right) \setminus \{k-2\}.$$

Then, for all $k \geq 2l$:

$$|L_k| = |L_{k-1}| + |L_{k-2}|.$$

Even more: $|L_k| = 2F_k$.

Proof In [5] it is proven that for $l = 2$ and $L_2 = \{1, 3\}$, the recursively defined sets L_k satisfy $|L_k| = 2F_k$ for all $k \geq 2$. This proves the lemma in the particular case in which $l = 2$ and $L_2 = \{1, 3\}$.

Next we will prove that if l, l' are integers and the multisets L_l, $L'_{l'}$ satisfy the hypothesis, then $L_k = L'_k$ for all $k \geq \max(2l, 2l')$. This, together with the result in [5] will end the proof.

Suppose $m \in L_s$, $m \neq s+1$. Then m gives rise to a subset $\{0, \ldots, m-1\} \subseteq L_{s+1}$ and to a subset in L_{s+2} whose maximum element is $m-2$ and to a subset in L_{s+3} whose maximum element is $m-3$ and so on. However, the fact that $m \in L_s$ does not affect $L_{s'}$ for $s' > k+m$. Similarly, the only element in L_l that affects L_k for any $k \geq 2l$ is $l+1$. Consequently, $L_k = L'_k$ for any $k \geq 2l$. □

A rough idea of future approaches to the Fibonacci-like problem would be: observe that the number of strong generators becomes negligible compared to all effective generators as $g \to \infty$, then the semigroup tree behaves more and more like the tree A from [5]. So roughly speaking we are in the situation of Proposition 13. Pushing this idea further could help to solve the Fibonacci-like conjecture.

Finally, we would like to mention some computational evidence that suggests that strong generators appear quite regularly. Let n_g^i be the number of numerical semigroups of genus g with i strong generators. Then we conjecture that

$$n_g^i = 0 \quad \text{for } i > \left\lfloor \frac{g-1}{2} \right\rfloor.$$

It is observed that as g increases, $n_g^{\lfloor \frac{g-1}{2} \rfloor - j}$ approaches a constant for g even and another constant for g odd. So, we can define two sequences

$$e_j = \lim_{k \to \infty} n_{2k}^{k-1-j},$$

$$o_j = \lim_{k \to \infty} n_{2k+1}^{k-j}.$$

The first terms of the sequence e have been observed to be

$$2, 2, 5, 12, 21, 45.$$

And the first terms of the sequence o have been observed to be

$$1, 2, 3, 8, 14, 34 - 35.$$

It seems that $e_j \geq \sum_{l=0}^{j-1} e_l$ and the same for o, so we conjecture in particular that the e- and o-sequences are superincreasing.

5 Conclusions

In this paper we went a step further on the study of the structure of the semigroup tree. Namely we described the nodes that correspond to some well-studied classes of numerical semigroups, like symmetric, pseudo-symmetric and Arf. Apart from this we also considered what kind of chains appear in the semigroup tree. Namely, when a node (semigroup) belongs to an infinite chain, and when the number of such chains is finite/infinite. We concluded the paper with some conjectures and observations regarding the number of strong generators. These conjectures hopefully can help in tackling the Fibonacci-like problem.

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