Computers and Mathematics with Applications 58 (2009) 1433–1440

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

On reliability indices of communication networks

Juan Alberto Rodríguez-Velázquez [a,*], Aida Kamišalić [a,b], Josep Domingo-Ferrer [a]

[a] Universitat Rovira i Virgili, Department of Computer Engineering and Mathematics, UNESCO Chair in Data Privacy, Av. Països Catalans 26, 43007 Tarragona, Catalonia, Spain

[b] University of Maribor, Faculty of Electrical Engineering and Computer Science, Smetanova 17, 2000 Maribor, Slovenia

ARTICLE INFO

Article history:
Received 5 September 2008
Received in revised form 29 June 2009
Accepted 2 July 2009

Keywords:
Wiener index
Information theory
Graph eigenvalues

ABSTRACT

The aim of this paper is to study the global reliability of communication networks. We assume that, in a communication network, the weights of the edges quantify the volume or the quality of the information transmitted by the nodes. In such a case, the strength of a path (resp. walk), called the *reliability* of the path (resp. walk) can be calculated as the product of the weights of the edges belonging to the paths (resp. walks). We introduce three indices to compute the reliability of a digraph (resp. graph). The first one is a version of Wiener index where we consider only the most reliable path between each pair of nodes. The second notion of reliability index considers reliability of all walks between each pair of nodes instead of taking into account only the most reliable path. The last one is a generalization of the functional centralization to the case of weighted networks. In this case, the notion of reliability index considers, for each node, the reliability of all closed walks starting and ending in the node. In addition, we propose a method for computing the introduced indices. Application of some of the proposed indices to trust-weighted social networks is also discussed.

1. Introduction

Various measurable properties of networks are usually expressed by means of numbers. In order to link network topology to any real network property, one must first convert the information contained in the network into a numerical characteristic. Every number uniquely determined by the structure of a graph is called a *graph invariant*.

Those graph invariants which are used for the study of the structural properties of networks are usually called *topological indices*. Among the most extensively studied topological indices, we cite the Wiener index [1] and the Randić index [2]. In some cases, according to the nature of the structural characteristic studied in a network, it is necessary to study invariants defined on weighted graphs. For instance, the Randić index of a graph $G = (V, E)$, defined by $R(G) = \sum_{uv\in E}[d(u)d(v)]^{\frac{-1}{2}}$, can be seen as an index defined on a weighted graph whose edge weights are $w(u, v) = [d(u)d(v)]^{\frac{-1}{2}}$, where $d(u)$ and $d(v)$ denote the degree of the corresponding vertices.

In this paper, we are interested in the study of the global reliability of communication networks. We will not discuss here how to assign weights to edges, as we assume that the literature provides appropriate measures for the various applications of weighted graphs. We will only mention two brief examples for illustration, the second of which will be extended in Section 5 below. For instance, suppose that, in a communication network, the weights (w) of the edges reflect the quality of the information transmitted or the trust between nodes:

* Corresponding author.
E-mail addresses: juanalberto.rodriguez@urv.cat (J.A. Rodríguez-Velázquez), aida.kamisalic@urv.cat (A. Kamišalić), josep.domingo@urv.cat (J. Domingo-Ferrer).

0898-1221/$ – see front matter
doi:10.1016/j.camwa.2009.07.019

1. The probability that a bit transmitted by node *i* is erroneously received by node *j* is a simple way to quantify the quality of transmission between node *i* and node *j*. This probability can be viewed as the weight $w(i, j)$ of edge (i, j).
2. The level of trust between node *i* and node *j* in a social network can be quantified as a weight $w(i, j) \in [0, 1]$ proportional to how much node *i* trusts node *j*. Extreme cases are no trust ($w(i, j) = 0$) and maximum trust ($w(i, j) = 1$). See Section 5 below for further details on the application to social networks.

Let a and b be vertices of a simple graph G. An *ab-walk* of length k between a and b is a sequence $a = v_0, v_1, \ldots, v_k = b$ of vertices such that v_i is adjacent to v_{i+1} for $i = 0, \ldots, k-1$. An *ab-path* is an *ab-walk* in which no vertex is repeated. In general, we are interested in the cases where the weight of a path (resp. walk) can be computed as the product of the weights of the edges belonging to it. The weight of the path (resp. walk) is called its *reliability*. More formally, let $G = (V, E, w)$ be a weighted digraph where $w : E \mapsto (0, 1]$ is the weight function of G. For a path (resp. walk) $P : u = v_i, v_{i+1}, \ldots, v_k = v$ in G we define the *reliability of* P as

$$w(P) = \prod_{l=i}^{k-1} w(v_l, v_{l+1}).$$

Let us assume that several routes exist from one node to another. The maximum reliability between two nodes (in terms of transmission quality, trust, etc.) is reached using the path with maximum reliability among those connecting both nodes. Inspired by this idea, we introduce the notion of reliability of a digraph (resp. graph) G according to the reliability of its paths or walks.

1.1. Contribution and plan of this article

We introduce three indices to compute the reliability of a digraph (resp. graph). The first one is described in Section 2 and is a version of Wiener index [1] where we consider only the most reliable path between each pair of nodes. The second notion of reliability index, described in Section 3, considers reliability of all walks between each pair of nodes instead of taking into account only the most reliable path. The last index, described in Section 4, is a generalization of the functional centralization [3] to the case of weighted networks. In this case, the notion of reliability index considers, for each node, the reliability of all closed walks starting and ending in the node. Section 5 discusses the application of some of the presented indices to characterize social networks with trust-weighted relationships. Conclusions are drawn in Section 6.

2. Reliability Wiener index

The *Wiener index* $W(G)$ of a graph G with vertex set $\{v_1, v_2, \ldots, v_n\}$ defined as the sum of distances between all pairs of vertices of G,

$$W(G) := \frac{1}{2} \sum_{i=1,j=1}^{n} \partial(v_i, v_j),$$

is the first mathematical invariant reflecting the topological structure of a molecular graph.

This topological index has been extensively studied. For instance, a comprehensive survey on the direct calculation, applications and the relation of the Wiener index of trees with other parameters of graphs can be found in [1]. Moreover, a list of 120 references of the main works on the Wiener index of graphs can be found in the referred survey.

Alternatively, the Wiener index can be defined as

$$W(G) = \frac{1}{2} \sum_{v \in V(G)} S(v),$$

where $S(v)$ denotes the *status* of the vertex v [4]:

$$S(v) := \sum_{u \in V(G)} \partial(u, v).$$

Let $G = (V, E, w)$ be a weighted digraph where $w : E \mapsto (0, 1]$ is the weight function of G.

For two vertices $i, j \in V$, we denote by $P_{\vec{ij}}$ the set of all directed paths from i to j. We denote by $F_{\vec{ij}}$ the weight of the most reliable path from i to j:

$$F_{\vec{ij}} := \max_{P \in P_{\vec{ij}}} \{w(P)\}. \tag{1}$$

So, we say that $F_{\vec{ij}}$ is the *reliability* of (i, j).

We define the *out-reliability* $R_1^+(i)$ of a vertex i in a digraph G of n vertices by

$$R_1^+(i) := \sum_{j=1}^{n} F_{\vec{ij}}. \tag{2}$$

The local index R_1^+ imposes a good ranking according to the capacity of transmitting reliable "information" to all other actors, where the information is transmitted through the most reliable path.

We define the *out-reliability Wiener index of G* by

$$W_{R_1^+}(G) := \sum_{v \in V(G)} R_1^+(v). \tag{3}$$

The out-reliability Wiener index of G is a measure of the capacity of the vertices of G of transmitting information in a reliable form, where the information is transmitted through the most reliable path.

Notice that we can define an analogous index $W_{R_1^-}(G)$ in order to measure the capacity of the vertices of G of receiving information in a reliable form. Such index is based on the *in-reliability* of a vertex i defined as

$$R_1^-(i) := \sum_{j=1}^{n} F_{\vec{ji}}. \tag{4}$$

Obviously, in the case of a graph G, $R_1^- = R_1^+$ and, as a consequence, $W_{R_1^+}(G) = W_{R_1^-}(G)$. Therefore, in the case of graphs we define the reliability Wiener index by

$$W_{R_1}(G) := \frac{1}{2} \sum_{v \in V(G)} R_1(v),$$

where $R_1 := R_1^- = R_1^+$.

We should point out that

$$w(P) = \exp\left(\sum_{l=1}^{k} \ln\left(w(v_i, v_{i+1})\right)\right).$$

As a consequence, the problem of finding $F_{\vec{ij}}$ can be solved as follows:

$$R_1^+(i) = \sum_{j=1}^{n} \exp(-l_{\vec{ij}}),$$

where

$$l_{\vec{ij}} := \min_{P \in P_{\vec{ij}}} \{w'(P)\}$$

and

$$w'(P) := -\sum_{l=1}^{k} \ln\left(w(v_i, v_{i+1})\right).$$

We conclude that the most reliable path from i to j in $G = (V, E, w)$ can be calculated by Dijkstra's algorithm on a weighted digraph $G' = (V, E, w')$ where the weight function $w' : E \to \mathbb{R}_+$ is defined by $w'(i, j) := -\ln(w(i, j))$. The reliability of (i, j) in $G = (V, E, w)$ is

$$F_{\vec{ij}} = \exp(-l_{\vec{ij}}).$$

Moreover, $l_{\vec{ij}}$ can be calculated by Floyd's (or Dijkstra's) algorithm on the weighted digraph $G' = (V, E, w')$.

One advantage of the reliability Wiener index is that it is applicable in the case of graphs and in the case of digraphs. However, it only takes into account the most reliable path between each pair of nodes. In general, nodes may use all of the paths (or walks) connecting them, rather than just the most reliable ones. For instance, suppose that two nodes want to start a relationship, but the most reliable path between them is unexpectedly blocked. If there exists another path or walk, the two nodes are likely to use it, even if it is longer and less reliable. The reliability index proposed in the following section takes into account all walks between each pair of nodes.

3. Reliability index R_*

As in the case of paths, for a walk $P : u = v_i, v_{i+1}, \ldots, v_k = v$ we define the *reliability of P* as

$$w(P) = \prod_{l=i}^{k-1} w(v_l, v_{l+1}).$$

We denote by $W_k(\vec{ij})$ the set of all walks of length k from i to j. We denote by $\mu_k(\vec{ij})$ the reliability of all walks of length k from i to j:

$$\mu_k(\vec{ij}) = \sum_{P \in W_k(\vec{ij})} w(P).$$

A digraph G is *strongly connected* if between every pair of distinct vertices i and j in G there is a directed path of finite length that begins at i and ends at j.

We define the *out-reliability* $R_*^+(i)$ of i in a strongly connected weighted digraph G by

$$R_*^+(i) := \sum_{j=1, j\neq i}^{n} \sum_{k=0}^{\infty} \mu_k(\vec{ij}).$$

We define the index R_* of G by

$$R_*(G) := \sum_{v\in V(G)} R_*^+(v).$$

This index is a measure of the capacity of the vertices of G of transmitting information reliably, where the information is transmitted through all possible walks from each vertex to others.

Now we are going to establish sufficient conditions for the convergence of the series $\sum_{k=0}^{\infty} \mu_k(\vec{ij})$. In addition, we need to obtain a method for computing $R_*(G)$.

If $G = (V, E, w)$ is a weighted digraph, the entry a_{ij} of the adjacency matrix $\mathbf{A} = (a_{ij})$ is the weight of the edge $(i, j) \in E$. Hence, $a_{ij} > 0$ if the vertex i is adjacent to j and $a_{ij} = 0$ otherwise.

Theorem 1. *Let* $A = (a_{ij})$ *be the adjacency matrix of a strongly connected digraph* G. *If* $\max_{i,j}\{a_{ij}\} \leq \frac{1}{n}$, *then* $R_*(G) = \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \left((I-A)^{-1}\right)_{ij}$.

Proof. It is well-known that if i and j are vertices of a graph G and $\mathbf{A}$ is the adjacency matrix of G, then the number of walks of length k in G, from i to j, is the entry in position (i, j) of the matrix $\mathbf{A}^k$. In the case of a strongly connected weighted digraph G we have

$$\mu_k(\vec{ij}) = \left(\mathbf{A}^k\right)_{ij}. \tag{5}$$

By (5), $\sum_{k=0}^{\infty} \mu_k(\vec{ij}) = \sum_{k=0}^{\infty} \left(\mathbf{A}^k\right)_{ij}$. Moreover, it was shown in [5] that for a matrix A of order $n \times n$, the matrix $(I - A)$ is invertible if there is a matrix norm $\|\cdot\|$ such that $\|A\| < 1$. If this condition is satisfied,

$$\sum_{k=0}^{\infty} A^k = (I - A)^{-1}.$$

By using the maximum column sum matrix norm $\|\cdot\|_1$,

$$\|A\|_1 \equiv \max_j \sum_{i=1}^{n} |a_{ij}|,$$

we obtain

$$\max_{i,j}\{a_{ij}\} \leq \frac{1}{n} \Rightarrow R_*^+(i) = \sum_{j=1, j\neq i}^{n} \left((I-A)^{-1}\right)_{ij}. \quad \square$$

As can be seen, this index is easy to compute, but it has the drawback of being applicable only to networks with small weights, *i.e.*, $\max_{i,j}\{a_{ij}\} \leq \frac{1}{n}$. The index proposed in the following section is applicable to any weighted graph.

We should point out that the entries of $(I - A)^{-1}$ have been used in [6] in the study of the *path accessibility* to a vertex from other vertices.

4. Functional reliability index

In this section, $G = (V, E, w)$ denotes a weighted graph. We will use the above notation but considering the edges as non-ordered pairs. Since the adjacency matrix, $\mathbf{A}$, of G is a symmetric matrix with real entries, there exists an orthogonal matrix $U = (u_{ij})$ such that $\mathbf{A} = UDU^T$ where $D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ whose diagonal entries are the eigenvalues of $\mathbf{A}$, and the columns of U are the corresponding eigenvectors that form an orthonormal basis of the Euclidean space $\mathbb{R}^n$. It must be emphasized that, if the graph G is connected, then the symmetric and non-negative matrix $\mathbf{A}$ is irreducible. As a consequence, the main eigenvalue of $\mathbf{A}$ has a positive eigenvector of multiplicity one. This fact facilitates the use of the main eigenvector as a measure of centrality in graphs.

It follows that the reliability of all walks of length k in G, from i to j, is

$$\mu_k(ij) = \left(\mathbf{A}^k\right)_{ij} = \sum_{s=1}^{n} u_{is}u_{js}\lambda_s^k.$$

Moreover, the reliability of all closed walks of length k starting and ending in vertex i in G is given by the local spectral moments $\mu_k(i)$, which are simply defined as the ith diagonal entry of the kth power of the adjacency matrix $\mathbf{A}$:

$$\mu_k(i) = \left(\mathbf{A}^k\right)_{ii} = \sum_{s=1}^{n} (u_{is})^2 \lambda_s^k. \tag{6}$$

Let λ be the spectral radius[1] of G. Let f be a function, $f : \mathbb{R} \mapsto \mathbb{R}$, whose Taylor series is $f(x) = \sum_{k=0}^{\infty} \alpha_k x^k$, $|x| < \lambda_*$, where $\lambda_* > \lambda$. We define the *functional reliability* of i, $R_f(i)$, as

$$R_f(i) := \sum_{j=0}^{\infty} \alpha_j \mu_j(i).$$

In this case, the reliability of all closed walks of length l is weighted by α_l. Thus, we can select the function f according to the features that we need to measure. Some interesting cases will be explained afterwards. Note that $R_f(i)$ is a generalization to weighted graphs of the functional centrality of i introduced in [3].

We define the *functional reliability index* of G by

$$R_f(G) := \sum_{i=1}^{n} R_f(i).$$

We remark that, in the case of unweighted graphs, $R_f(G)$ coincides with the functional centralization of G introduced in [3]. Moreover, Theorem 2 (and its proof) is completely analogous to the previous one obtained in [3] on the functional centralization. Even so, we include here the proof of Theorem 2 for completeness.

For any $i \in V$ we denote by $\ell^i(\mathbb{N})$ the space of real sequences $y = (y_j)_{j=0}^{\infty}$ such that

$$\sum_{j=0}^{\infty} y_j \mu_j(i) < \infty.$$

Theorem 2. *Let $G = (V, E)$ be a simple graph of order n. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of G. Let $\lambda_* > \lambda_1$ and let f be a real function such that $f(x) = \sum_{k=0}^{\infty} \alpha_k x^k$, for $|x| < \lambda_*$. Then for each $i \in V$, $a = (\alpha_0, \alpha_1, \ldots, \alpha_k, \ldots) \in \ell^i(\mathbb{N})$ and the functional reliability index of G is*

$$R_f(G) = \sum_{j=1}^{n} f(\lambda_j).$$

Proof. As above, let $U = (u_{ij})$ denote an orthogonal matrix whose columns are eigenvectors of G. By the definition of $R_f(i)$ and (6), we obtain

$$R_f(i) = \sum_{k=0}^{\infty} \alpha_k \left(\sum_{j=1}^{n} \lambda_j^k (u_{ij})^2 \right). \tag{7}$$

On the other hand, Series (7) is obtained by adding term by term the following convergent series:

$$(u_{i1})^2 \sum_{k=0}^{\infty} \alpha_k \lambda_1^k = (u_{i1})^2 f(\lambda_1)$$

$$(u_{i2})^2 \sum_{k=0}^{\infty} \alpha_k \lambda_2^k = (u_{i2})^2 f(\lambda_2)$$

$$\vdots$$

$$(u_{in})^2 \sum_{k=0}^{\infty} \alpha_k \lambda_n^k = (u_{in})^2 f(\lambda_n).$$

Thus, Series (7) converges to $\sum_{j=1}^{n} (u_{ij})^2 f(\lambda_j)$. As a consequence, the result follows. □

We should point out that the results of this section are also applicable to a weighted digraph or a weighted multidigraph G if and only if the adjacency matrix of G is unitarily diagonalizable.

Suppose that the problem is to measure the reliability of the graph according to the reliability of closed walks of length $k \geq 2$ containing the vertex, *i.e.*, the local spectral moments. In this case we take the function $f(x) = x^k$ and, in consequence, we call this index *monomial reliability index*. Thus, the monomial reliability index $R_k(G)$ is

$$R_k(G) = \sum_{j=1}^{n} \lambda_j^k.$$

Suppose that the problem is to measure the reliability of the graph according to the reliability of closed walks of length less than or equal to k containing the vertex. In this case we take the function f as the polynomial of degree k,

[1] The largest eigenvalue of the adjacency matrix of G.

$q_k(x) = x^2 + x^3 + \cdots + x^k$, so that

$$R_{q_k}(G) = \sum_{j=1}^{n} \left(\lambda_j^2 + \lambda_j^3 + \cdots + \lambda_j^k\right).$$

It is well-known, there are graphs that do not have odd closed walks, *i.e.*, the bipartite graphs. On the other hand, it would be of interest to measure the reliability of the graphs according to the reliability of closed walks of odd (resp. even) length containing the vertex. In such case, we can take an odd (resp. even) function. For instance, the *odd reliability* of i is defined as

$$R_{\text{odd}}(i) := \frac{\mu_1(i)}{1!} + \frac{\mu_3(i)}{3!} + \frac{\mu_5(i)}{5!} + \cdots$$

Hence, in this case, $f(x) = \sinh(x)$ and the *odd reliability index* of G is

$$R_{\text{odd}}(G) = \sum_{j=1}^{n} \sinh(\lambda_j).$$

Analogously, the *even reliability index* is

$$R_{\text{even}}(G) = \sum_{j=1}^{n} \cosh(\lambda_j).$$

It would be of interest to measure the reliability of the graphs according to the reliability of the closed walks of any length containing the vertex. In this case, the reliability of all closed walks of a given length is appropriately weighted such that their influence on the index decreases as the length of the walk increases. Thus we can take $f(x) = \exp(x)$ and the reliability of G is

$$R_e(G) = \sum_{j=1}^{n} \exp(\lambda_j).$$

The R_e index is the generalization to weighted graphs of the *EE* index used for ranking proteins according to their degree of folding [7].

5. Application to social networks with trust-weighted relationships

In the above sections, several applications of some of the above indices have been hinted at: reliability assessment, transmission quality assessment, centrality assessment in social networks, protein ranking. In this section, we will discuss, in some detail, how the reliability Wiener index can be used to characterize trust-weighted social networks.

Social networks have become an important web service [8] with a broad range of applications: collaborative work, collaborative service rating, resource sharing, searching for new friends etc. They have become an object of study both in computer and social sciences, even with dedicated journals and conferences. They can be defined as a community of web users where each network user can publish and share information and services (personal data, blogs and, in general, resources). In some social networks, users can specify how much they trust other users, by assigning them a trust level [9,10]. It is also possible to establish several types of relationships among users (for example, "colleague of", "friend of", etc.). The trust level and the type of relationship are used to decide whether access is granted to resources and services being offered.

In Section 5.1 we give some background on trust-weighted social networks, with a focus on those offering private relationships. In Section 5.2 we discuss the meaning of the reliability Wiener index in those networks.

5.1. Background on trust-weighted social networks

We will give some details here on the decision process followed by a resource owner in a trust-weighted social network to decide whether access is to be granted to a resource requestor. Illustration will be on a recent and sophisticated kind of social networks, which not merely enforce secure access (only requestors sufficiently trusted by the owners are granted access) but also guarantee that relationships stay private. As pointed out in [11,12], the availability of information on relationships (trust level, relationship type) has increased with the advent of the Semantic Web and raises privacy concerns: knowing who is trusted by a user and to what extent discloses a lot about that user's thoughts and feelings. See [13] for an analysis of related abuses.

These privacy issues have motivated some social networks [14,15] to enforce simple protection mechanisms, according to which users can decide whether their resources and relationships should be public or restricted to themselves and those users with whom they have a direct relationship. Unfortunately, such straightforward mechanisms result in overly restrictive policies.

In [16], a more flexible access control scheme is described, whereby a *requestor* can be authorized to access a resource even if he has no direct relationship with the *resource owner*, but he is within a specified depth in the relationship graph. *Access rules* are used, which specify the set of *access conditions* under which a certain resource can be accessed. Access conditions are a function of the relationship type, depth and trust level. Relationship certificates based on symmetric-key cryptography are used by a requestor to prove that he satisfies some specific access conditions. To access resources held by a node with whom the requestor has no direct relationship, the requestor retrieves, from a central node, the chain of relationship certificates along the path from the resource owner to himself. Clearly, the central node is a trusted third party, as it knows the relationships of all nodes in the network.

An innovative privacy-preserving approach is described in [11] which leans on the access model in [16] and focuses on relationship protection: a user can keep private that he has a relationship of a given type and trust level with another user. In [11] a rather complex scheme is proposed to manage relationship certificates. Encrypted certificates are stored at a central node; due to encryption, the central node does not have access to the cleartext certificates, so it does not need to be trusted in this respect. However, the central node needs to be trusted in the following aspects: (i) trust level computation when several relationship certificates are chained (indirect relationship between a resource requestor and a resource owner); (ii) certificate revocation enforcement when a relationship ceases to exist (the central node must maintain a certificate revocation list and inform the other nodes about new revocations).

In [17] a protocol is proposed which overcomes the shortcomings detected in [11]. Specifically, the author presents a public-key protocol which achieves relationship protection without the presence of a central node working as Trusted Third Party (TTP). In addition to that, this protocol avoids revealing the content of relationships to the resource requestor and substantially simplifies relationship revocation. Some problems remain in this scheme: (i) the resource owner learns the relationships, and their trust level, between the users who collaborate in resource access; (ii) as a consequence of the previous problem (or due to lack of relationships), a requestor may be unable to find intermediate nodes willing to help him in resource access.

In [18], a new protocol is presented which offers the same features of [11] and [17] while providing a solution which addresses the drawbacks left open in [17]. The drawbacks are solved at the cost of assuming the existence of an optimistic TTP which only acts in case of conflict between the users of the social network. Such authority is not needed during the normal network execution. Therefore, this solution performs better than a (non-optimistic) TTP mediating all access requests. The proposed scheme prevents the resource owner from learning the relationships and the trust levels between the users who collaborate in the resource access. In this way, the privacy threat detected in [17] is solved and the number of users who might refuse collaboration due to privacy concerns is minimized. As a result, the chances for certain nodes to become isolated at certain periods of time are reduced.

5.2. Reliability indices and social networks

In [17,18], the trust level between a resource requestor and a resource owner is computed based on the relationship path between them yielding the highest trust level. The resource owner uses the computed trust level to decide whether access is to be granted to the requestor. If we take the trust level between two nodes as the weight of the edge connecting them, the trust level between requestor j and owner i can be measured as the reliability of (i, j), that is $F_{\vec{ij}}$, defined in Expression (1) (see Section 2 above).

The out-reliability $R_1^+(i)$ (Expression (2) above) thus measures the overall willingness (based on trust) of owner i for sharing his resources with the rest of nodes in the social network.

For a social network SN, its out-reliability Wiener index $W_{R_1^+}(SN)$ (Expression (3) above) measures the overall "generosity" of nodes in the network, that is, their willingness (based on trust) for sharing their resources with the rest of nodes. A high value of $W_{R_1^+}(SN)$ indicates that SN consists of confident, generous nodes. A low value indicates a very loose network consisting of mistrusting nodes.

Similarly, the in-reliability $R_1^-(i)$ (Expression (4) above) measures the confidence inspired by node i to the rest of nodes. If node i has a very high (resp. low) in-reliability, it will be very easy (resp. difficult) for i to gain access to resources offered by other nodes. As to the in-reliability Wiener index $W_{R_1^-}(SN)$ for the overall social network, it measures "how easy is life" in that network for requestors, that is, how easy is for them to get access to resources.

In order to include out- and in-reliability indices in the social network framework proposed in [18], each node i should send to the optimistic TTP $F_{\vec{ij}}$ for every other node j. Those edge reliabilities can be obtained by *ad hoc* computation (in the same way owners compute their trust level vs requestors) or be computed by node i when node j requests access to a resource owned by i. The latter option involves no overhead (because i must compute $F_{\vec{ij}}$ anyway to decide on access) but assumes that *every node* is a resource owner whose resources which will of interest to some requestor (otherwise some edge reliabilities will never be computed).

In this way, the optimistic TTP can compute the out- and in-reliabilities for each node. The TTP can then compute the out- and in-reliability Wiener indices. While the node out- and in-reliabilities must probably be kept confidential, the out- and in-reliability Wiener indices for the overall social network can be usefully published without encroaching on the privacy of particular nodes. Publication of the current Wiener indices of a social network can help people deciding whether to enter or quit the network. *If a social network has low Wiener indices, it is a rather hostile network which people might wish to quit. On the other hand, a social network with high Wiener indices is very attractive for new people to join.*

Note. Relying on the TTP to compute the Wiener indices does not change its optimistic nature specified in [18]. Indeed, the computation of Wiener indices does not imply that the TTP should mediate every access request.

6. Conclusions

We have presented three indices for evaluating the reliability in a digraph or a graph. The first of them is a version of the Wiener index considering the most reliable path between each pair of nodes. The second considers all walks between each pair of nodes. The third one is a generalization of the functional centralization to the case of weighted networks.

While their primary application is the assessment of reliability and/or transmission quality in communication networks, other applications exist. Indeed, the functional reliability index can be used to rank proteins according to their degree of folding, and the Wiener indices are useful in social networks, both to assess the centrality of nodes and the overall friendliness (in terms of trust to provide resource access) of the network.

Acknowledgments

This work was partly supported by the Spanish Government through projects TSI2007-65406-C03-01 ''E-AEGIS'' and CONSOLIDER INGENIO 2010 CSD2007-00004 ''ARES'', and by the Government of Catalonia through grant 2009 SGR 1135. The authors are with the UNESCO Chair in Data Privacy, but their views do not necessarily reflect the position of UNESCO nor commit that organization. The third author is partly supported as an ICREA Acadèmia researcher by the Government of Catalonia.

References

[1] A.A. Dobrynin, R. Entringer, I. Gutman, Wiener index of trees: Theory and applications, Acta Applicandae Mathematicae 66 (2001) 211–249.
[2] M. Randić, On the characterization of molecular branching, Journal of the American Chemical Society 97 (1975) 6609–6615.
[3] J.A. Rodríguez, E. Estrada, A. Gutiérrez, Functional centrality in graphs, Linear and Multilinear Algebra 55 (3) (2007) 293–302.
[4] F. Buckley, F. Harary, Distance in Graphs, Addison-Wesley, Redwood City, CA, 1990.
[5] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, MA, 1990.
[6] P. Chebotarev, E.V. Shamis, On proximity measures for graph vertices, Automation and Remote Control 59 (10) (1998) 1443–1459.
[7] E. Estrada, Characterisation of the amino-acids contributions to the folding degree of proteins, Proteins: Structure, Function and Bioinformatics 54 (2004) 727–737.
[8] S. Staab, P. Domingos, P. Mika, J. Golbeck, L. Ding, T.W. Finin, A. Joshi, A. Nowak, R.R. Vallacher, Social networks applied, IEEE Intelligent Systems 20 (1) (2005) 80–93.
[9] R. Ashri, S.D. Ramchurn, J. Sabater, M. Luck, N.R. Jennings, Trust evaluation through relationship analysis, in: 4th International Joint Conference on Autonomous Agents and Multiagent Systems, ACM, 2005, pp. 1005–1011.
[10] J. Sabater-Mir, Towards the next generation of computational trust and reputation models, in: Modeling Decisions for Artificial Intelligence-MDAI, in: LNCS, vol. 3885, Springer-Verlag, 2006, pp. 19–21.
[11] B. Carminati, E. Ferrari, A. Perego, Private relationships in social networks, in: Private Data Management, IEEE Press, 2007.
[12] A.J. Mikroyannidis, Towards a social semantic web, Computer 40 (11) (2007) 113–115.
[13] S.B. Barnes, A privacy paradox: Social networking in the United States, First Monday 11 (9) (2006).
[14] Facebook, http://www.facebook.com.
[15] Videntity, http://videntity.org.
[16] B. Carminati, E. Ferrari, A. Perego, Rule-based access control for social networks, in: OTM Workshops, in: LNCS, vol. 4278, Springer-Verlag, 2006, pp. 1734–1744.
[17] J. Domingo-Ferrer, A public-key protocol for social networks with private relationships, in: Modeling Decisions for Artificial Intelligence, in: LNCS, vol. 4617, Springer-Verlag, 2007, pp. 373–379.
[18] J. Domingo-Ferrer, Alexandre Viejo, Francesc Sebé, Úrsula González-Nicolás, Privacy homomorphisms for social networks with private relationships, Computer Networks 52 (2008) 3007–3016.