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Integrability of the constrained rigid body

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Abstract The integrability theory for the differential equations, which describe the motion of an unconstrained rigid body around a fixed point is well known. When there are constraints the theory of integrability is incomplete. The main objective of this paper is to analyze the integrability of the equations of motion of a constrained rigid body around a fixed point in a force field with potential $U(\gamma) = U(\gamma_1, \gamma_2, \gamma_3)$. This motion subject to the constraint $\langle \nu, \omega \rangle = 0$ with ν is a constant vector is known as the Suslov problem, and when $\nu = \gamma$ is the known Veselova problem, here $\omega = (\omega_1, \omega_2, \omega_3)$ is the angular velocity and $\langle \, , \rangle$ is the inner product of $\mathbb{R}^3$.

We provide the following new integrable cases.

(i) The Suslov's problem is integrable under the assumption that ν is an eigenvector of the inertial tensor I and the potential is such that

$$U = -\frac{1}{2I_1 I_2}\left(I_1\mu_1^2 + I_2\mu_2^2\right),$$

where I_1, I_2, and I_3 are the principal moments of inertia of the body, μ_1 and μ_2 are solutions of the first-order partial differential equation

$$\gamma_3\left(\frac{\partial \mu_1}{\partial \gamma_2} - \frac{\partial \mu_2}{\partial \gamma_1}\right) - \gamma_2 \frac{\partial \mu_1}{\partial \gamma_3} + \gamma_1 \frac{\partial \mu_2}{\partial \gamma_3} = 0.$$

(ii) The Veselova problem is integrable for the potential

$$U = -\frac{\Psi_1^2 + \Psi_2^2}{2(I_1\gamma_2^2 + I_2\gamma_1^2)},$$

where Ψ_1 and Ψ_2 are the solutions of the first-order partial differential equation

$$(I_2 - I_1)\gamma_1\gamma_2\left\langle \gamma, \frac{\partial \Psi_2}{\partial \gamma}\right\rangle + I_1\gamma_2\frac{\partial \Psi_2}{\partial \gamma_1} - I_2\gamma_1\frac{\partial \Psi_2}{\partial \gamma_2}$$
$$- p\left(\gamma_3\left\langle \gamma, \frac{\partial \Psi_1}{\partial \gamma}\right\rangle - \frac{\partial \Psi_1}{\partial \gamma_3}\right) = 0,$$

where $p = \sqrt{I_1 I_2 I_3\left(\frac{\gamma_1^2}{I_1} + \frac{\gamma_2^2}{I_2} + \frac{\gamma_3^2}{I_3}\right)}$.

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Also it is integrable when the potential U is a solution of the second-order partial differential equation

$$2\frac{\partial U}{\partial \tau_3} + I_1 I_2 I_3 \frac{\partial^2 U}{\partial \tau_2^2} + (\tau_2 - I_1 - I_2 - I_3)\frac{\partial^2 U}{\partial \tau_3 \partial \tau_2} + \tau_3 \frac{\partial^2 U}{\partial \tau_3^2} = 0,$$

where $\tau_2 = I_1\gamma_1^2 + I_2\gamma_2^2 + I_3\gamma_3^2$ and $\tau_3 = \frac{\gamma_1^2}{I_1} + \frac{\gamma_2^2}{I_2} + \frac{\gamma_3^2}{I_3}$.

Moreover, we show that these integrable cases contain as a particular case the previous known results.

Keywords Ordinary differential equation · Invariant measure · Mechanical systems · Nonholonomic system · Constraint · Rigid body · Suslov problem · Veselova problem · Integrability

1 Introduction

For simplicity, we shall assume that all the functions which appear in this paper are of class $\mathcal{C}^\infty$, although most of the results remain valid under weaker smoothness.

Consider the differential system

$$\dot{\mathbf{x}} = \mathcal{X}(\mathbf{x}), \qquad x = (x_1, \ldots, x_N) \in \mathbb{R}^N. \tag{1}$$

Let Ω be an open and dense subset of $\mathbb{R}^N$. A nonconstant function $\Phi : \Omega \longrightarrow \mathbb{R}$ such that Φ is constant on the solutions of system (1) contained in Ω is called a *first integral*. We say that system (1) is *explicitly integrable* in Ω if it has $\Phi_k : \Omega \longrightarrow \mathbb{R}$ for $k = 1, \ldots, N-1$ functionally independent first integrals, i.e., the rank of the $(N-1) \times N$ Jacobian matrix

$$\frac{\partial(\Phi_1, \ldots, \Phi_{N-1})}{\partial(x_1, \ldots, x_N)},$$

is $N-1$ in all the points $(x_1, \ldots, x_N)$ of Ω except perhaps in a zero Lebesgue measure set.

Let Σ be an open subset of $\mathbb{R}^M$ and let $F_j : \Omega \times \Sigma \longrightarrow \mathbb{R}$ for $j = 1, \ldots, M$ be a smooth maps. The relation $F_j := F_j(\mathbf{x}, K_1, \ldots, K_M) = 0$, with $K_1, \ldots, K_M$ constants, is called a *general integral* of (1) if $\mathcal{X} F_j|_{F_1=\cdots=F_M=0} = 0$. System (1) is *implicitly integrable* if it admits $M = N-1$ general integrals $F_j = 0$, for $j = 1, \ldots, N-1$ such that the rank of $(N-1) \times (N-1)$ Jacobian matrix

$$\frac{\partial(F_1, \ldots, F_{N-1})}{\partial(K_1, \ldots, K_{N-1})}, \tag{2}$$

is $N-1$ in all the points $(\mathbf{x}, K_1, \ldots, K_{N-1})$ of $\Omega \times \Sigma$ except perhaps in a zero Lebesgue measure set. Indeed, under condition (2) and by the implicit function theorem, we can obtain from the set of $N-1$ general integrals $N-1$ local first integrals of the form $K_j = \Phi_j(\mathbf{x})$. Consequently, system (1) is *locally explicitly integrable* in view of the previous definition.

The integration theory of the differential equations, which describe the motion of mechanical systems with nonintegrable constraints (i.e., nonholonomic system) is not so complete as for the unconstrained systems (i.e., holonomic systems). This is due to several reasons. One of them is that the equations of motion of nonholonomic systems in general have no invariant measure, as they have the equations of motion of holonomic systems; see, for instance, [9].

The existence of an invariant measure simplifies the integration of the differential equations. It is well known the Euler–Jacobi theorem: If the differential system (1) has $N-2$ independent first integrals $\Phi_1, \ldots, \Phi_{N-2}$ and

$$\operatorname{div}\big(M(\mathbf{x})\mathcal{X}(\mathbf{x})\big) = \sum_{j=1}^{N} \frac{\partial(M(\mathbf{x})\mathcal{X}_j)}{\partial x_j} = 0,$$

for a convenient function $M(\mathbf{x}) > 0$, then the differential system is explicitly integrable. We observe that this condition using the divergence is necessary and sufficient for the existence of an invariant measure with respect to the action of the vector field (1) (due to Euler–Jacobi theorem, see for instance, [7, 9]).

Let $\bar{\Sigma}$ be an open subset of $\mathbb{R}^{N-2}$. System (1) is *quasi-implicitly integrable* in $\Omega \times \bar{\Sigma}$ if it has an invariant measure and admits $N-2$ general integrals $F_j(\mathbf{x}, K_1, \ldots, K_{N-2}) = 0$ for $j = 1, \ldots, N-2$ such that the rank of $(N-2) \times (N-2)$ Jacobian matrix

$$\frac{\partial(F_1, \ldots, F_{N-2})}{\partial(K_1, \ldots, K_{N-2})}$$

is $N-2$, in all the points $(x_1, \ldots, x_N)$ of Ω except perhaps in a zero Lebesgue measure set and for arbitrary constants $K_1, \ldots, K_{N-2}$.

Now we study the integrability theory for the motion of a rigid body around a fixed point. If this mechanical system is free of constraints, then its integrability it is well known (see, for instance, [1]). But the integration of the equations of motion of this mechanical system with constraints is incomplete. For example, the integrability is in general unknown when the constraint is of the form

$$\langle \nu, \omega\rangle = 0, \tag{3}$$

where $\nu = \nu(\gamma) = (\nu_1, \nu_2, \nu_3)$ is a vector of $\mathbb{R}^3$, γ is the unit vector of a spatially fixed axis in the coordinate system rigidly connected with the body and such that

$$\gamma = (\gamma_1, \gamma_2, \gamma_3) = (\sin z \sin x, \sin z \cos x, \cos z), \tag{4}$$

$(x, y, z) = (\varphi, \psi, \theta)$ are the Euler angles and $\omega = (\omega_1, \omega_2, \omega_3)$ is the angular velocity.

Applying the method of Lagrange multipliers, the equations of motion of the rigid body around a fixed point with the constraint (3) are (for more details, see for instance [1, 2])

$$\begin{aligned} I\dot{\omega} &= I\omega \wedge \omega + \gamma \wedge \frac{\partial U}{\partial \gamma} + \mu\nu, \\ \dot{\gamma} &= \gamma \wedge \omega, \quad \langle \nu, \omega\rangle = 0, \end{aligned} \tag{5}$$

where I is the inertial tensor of the rigid body, i.e.,

$$I = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{12} & I_{22} & I_{23} \\ I_{13} & I_{23} & I_{33} \end{pmatrix},$$

$U = U(\gamma_1, \gamma_2, \gamma_3)$ is the potential function, $\frac{\partial U(\gamma)}{\partial \gamma}$ is the gradient of $U(\gamma)$ with respect to γ, and $\wedge$ is the "wedge" product in $\mathbb{R}^3$.

We observe that the equations $\dot{\gamma} = \gamma \wedge \omega$ are known as the *Poisson differential equations.*

System (5) always has three independent first integrals, namely

$$\begin{aligned} &\Phi_1 = \langle \gamma, \gamma\rangle = \gamma_1^2 + \gamma_2^2 + \gamma_3^2, \qquad \Phi_2 = \langle \nu, \omega\rangle, \\ &\Phi_3 = \frac{1}{2}\langle I\omega, \omega\rangle + U(\gamma). \end{aligned} \tag{6}$$

We shall study two particular cases with the constraint (3): the Suslov problem ($\nu = \mathbf{a}$) and the Veselova problem ($\nu = \gamma$). The main objective of this paper is to present new cases of integrability for these two problems, which contains as particular cases the previous results on the integrability of these two problems.

1.1 The Suslov problem

The *Suslov problem* is the study of the motion of a rigid body around a fixed point and subject to the nonholonomic constraint $\langle \mathbf{a}, \omega\rangle = 0$, where $\mathbf{a}$ is a constant vector (see [14]). Suppose that the body rotates in a force field with potential $U(\gamma) = U(\gamma_1, \gamma_2, \gamma_3)$. Applying the method of Lagrange multipliers, the equations of motion are

$$\begin{aligned} I\dot{\omega} &= I\omega \wedge \omega + \gamma \wedge \frac{\partial U}{\partial \gamma} + \mu\mathbf{a}, \\ \dot{\gamma} &= \gamma \wedge \omega, \quad \langle \mathbf{a}, \omega\rangle = 0, \end{aligned} \tag{7}$$

System (7) always has the three independent first integrals (6) with $\nu = \mathbf{a}$.

In order to have real motions, we must take $\Phi_1 = 1$, $\Phi_2 = 0$ in (6). In this case, using the first integrals Φ_3 we can reduce the problem of integration of (7) to the problem of existence of an invariant measure and a fourth independent first integral Φ_4. Under these assumptions, by the Euler–Jacobi theorem (see, for instance, [3, 7]) the Suslov problem is integrable [10]. In general, system (7) has no invariant measure if the vector $\mathbf{a}$ is not an eigenvector of the tensor of inertia. The following result is well known, see [9].

Proposition 1 *If* $\boldsymbol{a}$ *is an eigenvector of the inertial tensor* I*, i.e.,*

$$I\boldsymbol{a} = \kappa\boldsymbol{a} \tag{8}$$

for some $\kappa \in \mathbb{R}$*, then the flow of system* (7) *preserves the Lebesgue measure in* $\mathbb{R}^6 = \mathbb{R}^3\{\omega\} \times \mathbb{R}^3\{\gamma\}$.

Suslov in [14] has considered the case when the body is in absence of external forces, i.e., $U = 0$. In this case, the system

$$I\dot{\omega} = I\omega \wedge \omega + \mu\mathbf{a},$$

or equivalently (see [5])

$$I\dot{\omega} = -\frac{\langle I\tilde{\mathbf{a}}, \omega\rangle}{\langle I^{-1}\tilde{\mathbf{a}}, \tilde{\mathbf{a}}\rangle}\left(I^{-1}\tilde{\mathbf{a}} \wedge \omega\right),$$

can be solved with respect to ω, i.e., it is integrable by quadratures. The analysis of theses quadratures shows that if (8) does not holds then all trajectories $\omega = \omega(t)$ approach asymptotically as $t \to \pm\infty$ to some fixed straight line on the plane $\langle \mathbf{a}, \omega \rangle = 0$. Consequently, Eq. (7) have no invariant measure. The question about the possibility to find $\gamma = \gamma(t)$ by quadratures in general remains open (for more details, see [5]).

If (8) holds, then along the solutions of (7) the kinetic moment $\langle I\omega, I\omega \rangle$ is a first integral (see [9]).

From now on, we suppose that equality (8) is fulfilled.

Without loss of generality, we can choose the vector $\mathbf{a}$ as the third axis, i.e., $\mathbf{a} = (0, 0, 1)$, and consequently the constrained becomes $\omega_3 = \dot{x} + \dot{y}\cos z = 0$. Then the equations of motion are

$$I_1\dot{\omega}_1 = \gamma_2\frac{\partial U}{\partial \gamma_3} - \gamma_3\frac{\partial U}{\partial \gamma_2}, \qquad I_2\dot{\omega}_2 = \gamma_3\frac{\partial U}{\partial \gamma_1} - \gamma_1\frac{\partial U}{\partial \gamma_2},$$
$$\dot{\gamma}_1 = -\gamma_3\omega_2, \qquad \dot{\gamma}_2 = \gamma_3\omega_1, \qquad \dot{\gamma}_3 = \gamma_1\omega_2 - \gamma_2\omega_1, \tag{9}$$

where $I = \mathrm{diag}(I_1, I_2, I_3)$, and I_k are the principal moments of inertia of the body.

Kharlamova–Zabalina in [8] studied the case when the body rotates in the homogenous force field with the potential $U = \langle \mathbf{b}, \gamma \rangle$ where the vector $\mathbf{b}$ is orthogonal to the vector $\mathbf{a}$. Under these conditions, the equations of motion have the first integral $\Phi_4 = \langle I\omega, \mathbf{b} \rangle$.

Kozlov consider the case when $\mathbf{b} = \lambda\mathbf{a}$, $\lambda \neq 0$. The integrability problem in this case was study in [9, 11]. If $I_1 \neq I_2$, apparently the equations have no additional first integral independent of the energy integral. When $I_1 = I_2$ and $U = \lambda\langle \mathbf{a}, \gamma \rangle$, there exists the fourth integral $\Phi_4 = \omega_1\gamma_1 + \omega_2\gamma_2$.

For the case $U = \lambda|I|\langle I^{-1}\gamma, \gamma \rangle$, where $|I| = \det I$, system (7) has the Klebsh–Tisserand first integral $\Phi_4 = \frac{1}{2}\langle I\omega, I\omega \rangle - \frac{1}{2}\lambda|I|\langle I^{-1}\gamma, \gamma \rangle$ (see, for instance, [9]).

Okuneva in [12] proved the integrability of the Suslov problem for the potential $U = \alpha\gamma_1 + \beta\gamma_2 + \frac{\lambda}{2}\langle I^{-1}\gamma, \gamma \rangle$, where α, β, and λ are constants. The first integral is $\Phi_4 = I_1\omega_1^2 - \lambda(I_2 - I_3)\gamma_2^2 - 2\beta\gamma_2$, or equivalently $\Phi_4 = I_2\omega_2^2 - \lambda(I_3 - I_1)\gamma_1^2 - 2\alpha\gamma_1$.

Dragovic et al. in [4] consider the case when the potential $U = c(\gamma_1, \gamma_2^2 + \gamma_3^2) - d(\gamma_2, \gamma_1^2 + \gamma_3^2)$ for arbitrary functions $c = c(\gamma_1, \gamma_2^2 + \gamma_3^2)$ and $d = d(\gamma_2, \gamma_1^2 + \gamma_3^2)$, proved that $\Phi_4 = \frac{1}{2}\langle I\omega, I\omega \rangle + I_2 c(\gamma_1, \gamma_2^2 + \gamma_3^2) - I_1 d(\gamma_2, \gamma_1^2 + \gamma_3^2)$ is a first integral of system (7).

1.2 The Veselova problem

The *Veselova problem* describes the motion of a rigid body, which rotates around a fixed point and subject to the nonholonomic constraint

$$\langle \gamma, \omega \rangle = \dot{y} + \dot{x}\cos z = 0. \tag{10}$$

Thus, in the case of the Veselova constraint, the projection of the angular velocity to a spatially fixed axis is zero.

Suppose that the body rotates in a force field with potential $U(\gamma_1, \gamma_2, \gamma_3)$. Applying the method of Lagrange multipliers, we write the equations of motion in the form

$$I\dot{\omega} = I\omega \wedge \omega + \gamma \wedge \frac{\partial U}{\partial \gamma} + \mu\gamma,$$
$$\dot{\gamma} = \gamma \wedge \omega, \quad \langle \gamma, \omega \rangle = 0, \tag{11}$$

where $I = \mathrm{diag}(I_1, I_2, I_3)$. System (11) has always three independent integrals (6) with $\nu = \gamma$.

As proved in [15], system (11) has an invariant measure with density

$$\sqrt{\frac{\gamma_1^2}{I_1} + \frac{\gamma_2^2}{I_2} + \frac{\gamma_3^2}{I_3}}.$$

Thus, in view of the Euler–Jacobi theorem, we obtain that if there exists a fourth first integrals Φ_4 independent with Φ_1, Φ_2, Φ_3, then the Veselova problem is integrable. In order to have real motions, we must take $\Phi_1 = 1$ and $\Phi_2 = 0$.

Remark 1 From (11), we get the equalities

$$\begin{aligned}
&\frac{d}{dt}(\gamma \wedge I\omega) \\
&\quad = \frac{d\gamma}{dt} \wedge I\omega + \gamma \wedge I\frac{d\omega}{dt} \\
&\quad = (\gamma \wedge \omega) \wedge I\omega + \gamma \wedge \left(I\omega \wedge \omega + \gamma \wedge \frac{\partial U}{\partial \gamma} + \lambda\gamma\right) \\
&\quad = (\gamma \wedge \omega) \wedge I\omega + (\omega \wedge I\omega) \wedge \gamma \\
&\qquad + \gamma \wedge \left(\gamma \wedge \frac{\partial U}{\partial \gamma}\right),
\end{aligned}$$

and by considering the identities

$$a \wedge (b \wedge c) = \langle a, c \rangle b - \langle a, b \rangle c,$$
$$a \wedge (b \wedge c) + b \wedge (c \wedge a) = -c \wedge (a \wedge b),$$

we obtain

$$\frac{d}{dt}(\gamma \wedge I\omega) = -\frac{\partial U}{\partial \gamma} + \gamma\left(\left\langle \gamma, \frac{\partial U}{\partial \gamma}\right\rangle - \langle I\omega, \omega\rangle\right). \quad (12)$$

From this relation, we deduce the equation

$$\frac{d(p\omega)}{dt} = \frac{1}{p}\left(\gamma \wedge I\frac{\partial U}{\partial \gamma} + I\gamma \wedge \gamma\left(\left\langle \gamma, \frac{\partial U}{\partial \gamma}\right\rangle - \langle I\omega, \omega\rangle\right)\right), \quad (13)$$

where $p = \sqrt{I_1 I_2 I_3(\frac{\gamma_1^2}{I_1} + \frac{\gamma_2^2}{I_2} + \frac{\gamma_3^2}{I_3})}$.

2 Statement of the main results

Our first main result is the following.

Theorem 1 *The motion of the rigid body subject to the nonholonomic constraint* $\omega_3 = 0$ *(Suslov problem), under assumption* (8) *is described by system* (9). *We suppose that the rigid body rotates under the action of a force field defined by the potential*

$$U = -\frac{1}{2I_1 I_2}\left(I_1\mu_1^2 + I_2\mu_2^2\right), \quad (14)$$

where $\mu_1 = \mu_1(\gamma_1, \gamma_2, \gamma_3, K_3, K_4)$ *and* $\mu_2 = \mu_2(\gamma_1, \gamma_2, \gamma_3, K_3, K_4)$ *with* K_3 *and* K_4 *constants. Assume that* μ_1 *and* μ_2 *are solutions of the first-order partial differential equation*

$$\gamma_3\left(\frac{\partial \mu_1}{\partial \gamma_2} - \frac{\partial \mu_2}{\partial \gamma_1}\right) - \gamma_2\frac{\partial \mu_1}{\partial \gamma_3} + \gamma_1\frac{\partial \mu_2}{\partial \gamma_3} = 0, \quad (15)$$

satisfying

$$\frac{\partial \mu_1}{\partial K_3}\frac{\partial \mu_2}{\partial K_4} - \frac{\partial \mu_2}{\partial K_3}\frac{\partial \mu_1}{\partial K_4} \neq 0 \quad \text{for all } (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}^3. \quad (16)$$

Then the following statements hold.

(a) *System* (9) *has the general integrals*

$$F_1 = I_1\omega_1 - \mu_2(\gamma_1, \gamma_2, \gamma_3, K_3, K_4) = 0,$$
$$F_2 = I_2\omega_2 + \mu_1(\gamma_1, \gamma_2, \gamma_3, K_3, K_4) = 0. \quad (17)$$

Moreover, system (9) *is quasi-implicitly integrable.*

(b) *The first integrals of Suslov, Kharlamova-Zabelina's, Kozlov, Dragović-Gajić-Jovanović, Klebsh-Tisserand, and Tisserand-Okuneva are particular cases of the first integrals of statement* (a).

(c) *Using* (17), *the Poisson equations for the Suslov problem take the form*

$$\dot\gamma_1 = -\gamma_3\frac{\mu_1}{I_2}, \qquad \dot\gamma_2 = -\gamma_3\frac{\mu_2}{I_1},$$
$$\dot\gamma_3 = -\gamma_1\frac{\mu_1}{I_2} - \gamma_2\frac{\mu_2}{I_1}. \quad (18)$$

We provide the solution $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$ *for all the cases of statement* (b).

The proof of Theorem 1 is given in Sect. 2.

We note that the potential of Eq. (14) contains as particular cases the potentials studied by the authors Suslov, Kharlamova-Zabelina's, Kozlov, Dragović-Gajić-Jovanović, Klebsh-Tisserand, and Tisserand-Okuneva, previously defined.

Remark 2 It is easy to check that the functions

$$\mu_1 = \frac{\partial \tilde S(\gamma_1, \gamma_2, \gamma_3, K_3, K_4)}{\partial \gamma_1} + \Psi_1\left(\gamma_2^2 + \gamma_3^2, \gamma_1, K_3, K_4\right),$$
$$\mu_2 = \frac{\partial \tilde S(\gamma_1, \gamma_2, \gamma_3, K_3, K_4)}{\partial \gamma_2} + \Psi_2\left(\gamma_1^2 + \gamma_3^2, \gamma_2, K_3, K_4\right), \quad (19)$$

are solutions of (15), where

$$\tilde S(\gamma_1, \gamma_2, \gamma_3, K_3, K_4) = S(\gamma_1, \gamma_2, K_3, K_4) + \int \Upsilon\left(\gamma_1^2 + \gamma_2^2, \gamma_3, K_3, K_4\right) d\left(\gamma_1^2 + \gamma_2^2\right),$$

and S, Ψ_1, Ψ_2, and Υ are arbitrary smooth functions for which (16) holds.

Remark 2 will be used for proving statement (b) of Theorem 1.

Note that in Theorem 1 we are working with two general integrals because we are only using the two first integrals $\Phi_1 = \langle \gamma, \gamma\rangle = 1$ and $\Phi_2 = \langle \omega, \gamma\rangle = 0$. There are two main reasons for working with two general integrals instead of using also the energy integral

$\Phi_3 = \langle I\omega, \omega\rangle + U(\gamma)$, which would allow to look for a unique general integral for determining the integrability of the Suslov problem. The first reason is that working with two general integrals all the computations for studying the integrable cases are easier. The second reason is that working with two general integrals, which are linear with respect to the angular velocity, the Poisson differential equations can be written in the form (18), which do not depend on the angular velocity, and consequently its integrability is in general easier, mainly in the cases of statement (b) (for more details, see [13]).

Remark 3 The solutions of (15) can be represented in a formal Laurent series

$$\mu_1 = \sum_{n,j,k\in\mathbb{Z}} a_{njk}\gamma_1^n\gamma_2^j\gamma_3^k,$$

$$\mu_2 = \sum_{n,j,k\in\mathbb{Z}} b_{njk}\gamma_1^n\gamma_2^j\gamma_3^k,$$

where $a_{njk} = a_{njk}(K_3, K_4)$ and $b_{njk} = b_{njk}(K_3, K_4)$ are coefficients, which satisfy the equalities

$$\begin{aligned} &j a_{njk} - (n+1)b_{n+1,j-1,k} \\ &\quad = (k+2)(a_{n,j-2,k+2} - b_{n-1,j-1,k+2}), \end{aligned}$$

where $n, j, k \in \mathbb{Z}$.

Our second result on the Veselova problem is the following.

Theorem 2 *The motion of the rigid body subject to the nonholonomic constraint* $\langle\gamma,\omega\rangle = 0$ (*Veselova problem*), *rotating under the action of a force field is defined by the potential*

$$U = -\frac{\Psi_1^2 + \Psi_2^2}{2(I_1\gamma_2^2 + I_2\gamma_1^2)}, \tag{20}$$

where $\Psi_j = \Psi_j(x, z, K_3, K_4)$ *for* $j = 1, 2$ *and* K_3 *and* K_4 *are constants. Assume that* Ψ_1 *and* Ψ_2 *are solutions of the first-order partial differential equation*

$$\begin{aligned} \Theta := {}& (I_1 - I_2)\sin z\cos z\sin x\cos x\frac{\partial\Psi_2}{\partial z} \\ &+ \left(I_1\cos^2 x + I_2\sin^2 x\right)\frac{\partial\Psi_2}{\partial x} \\ &- p\sin z\frac{\partial\Psi_1}{\partial z} = 0, \end{aligned} \tag{21}$$

satisfying

$$\frac{\partial\Psi_1}{\partial K_3}\frac{\partial\Psi_2}{\partial K_4} - \frac{\partial\Psi_2}{\partial K_3}\frac{\partial\Psi_1}{\partial K_4} \neq 0 \quad \textit{for all } (\gamma_1,\gamma_2,\gamma_3)\in\mathbb{R}^3. \tag{22}$$

Then the following statements hold.

(a) *System* (11) *has the general integrals*

$$\begin{aligned} &F_1 = I_1\omega_1\gamma_2 - I_2\omega_2\gamma_1 - \Psi_2 = 0, \\ &F_2 = p\omega_3 - \Psi_1 = 0. \end{aligned} \tag{23}$$

Moreover, system (11) *is quasi-implicitly integrable.*

(b) *If* $\Psi_1^2 + \Psi_2^2 = 2\Psi(x)$, *then system* (11) *has the first integral*

$$I_1\omega_1\gamma_2 - I_2\omega_2\gamma_1 = K_4.$$

Consequently the system is explicitly integrable.

(c) *If* $I_1 \neq I_2$ *and* $\Psi_1^2 + \Psi_2^2 = 2\Psi(\frac{I_1\gamma_2^2 + I_2\gamma_1^2}{\gamma_3^2})$, *then the system has the first integral*

$$\sqrt{I_1I_2I_3\left(\frac{\gamma_1^2}{I_1} + \frac{\gamma_2^2}{I_2} + \frac{\gamma_3^2}{I_3}\right)}\omega_3 = K_4.$$

If $I_1 = I_2$ *and* $\Psi_1^2 + \Psi_3^2 = 2\Psi^2(z)$, *then system has the first integral*

$$\sqrt{I_3\left(\frac{\gamma_1^2 + \gamma_2^2}{I_1} + \frac{\gamma_3^2}{I_3}\right)}\omega_3 = K_4. \tag{24}$$

Consequently, the system is explicitly integrable.

(d) *Using* (23), *the constraint* $\gamma_1\omega_1 + \gamma_2\omega_2 + \gamma_3\omega_3 = 0$, *and the fact that* $\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$, *the Poisson equations for the Veselova problem take the form*

$$\begin{aligned} \dot\gamma_1 &= -\frac{\gamma_2(I_1 + (I_2 - I_1)\gamma_1^2)\Psi_1 + \gamma_3\gamma_1\Psi_2}{p(I_2\gamma_1^2 + I_2\gamma_2^2)}, \\ \dot\gamma_2 &= \frac{\gamma_1(I_2 + (I_1 - I_2)\gamma_2^2))\Psi_1 - \gamma_3\gamma_2\Psi_2}{p(I_2\gamma_1^2 + I_2\gamma_2^2)}, \\ \dot\gamma_3 &= \frac{\gamma_1\gamma_2\gamma_3(I_1 - I_2)\Psi_1 + p(\gamma_2^2 + \gamma_1^2)\Psi_2}{p(I_2\gamma_1^2 + I_2\gamma_2^2)}. \end{aligned} \tag{25}$$

We observe that the integral (24) is well known (see, for instance, [4]).

Corollary 1 *Equation* (21) *is equivalent to the equation*

$$(I_2 - I_1)\gamma_1\gamma_2\left\langle \gamma, \frac{\partial \Psi_2}{\partial \gamma}\right\rangle + I_1\gamma_2\frac{\partial \Psi_2}{\partial \gamma_1} - I_2\gamma_1\frac{\partial \Psi_2}{\partial \gamma_2} - p\left(\gamma_3\left\langle \gamma, \frac{\partial \Psi_1}{\partial \gamma}\right\rangle - \frac{\partial \Psi_1}{\partial \gamma_3}\right) = 0. \quad (26)$$

The proof of Theorem 2 and Corollary 1 are given in Sect. 4.

We note that the potential (20) contains as particular subcases the potential studied by Veselova and Fedorov–Jovanovic [6].

Under the assumption $\Delta = (I_1 - I_2)(I_2 - I_3) \times (I_3 - I_1) \neq 0$, we introduce the coordinates (τ_1, τ_2, τ_3) such that

$$\tau_1 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2, \qquad \tau_2 = I_1\gamma_1^2 + I_2\gamma_2^2 + I_3\gamma_3^2, \qquad \tau_3 = \frac{\gamma_1^2}{I_1} + \frac{\gamma_2^2}{I_2} + \frac{\gamma_3^2}{I_3}. \quad (27)$$

Hence,

$$\begin{aligned} \gamma_1^2 &= r_1\big((I_2 + I_3)\tau_1 - \tau_2 - I_2I_3\tau_3\big),\\ \gamma_2^2 &= r_2\big((I_1 + I_3)\tau_1 - \tau_2 - I_1I_3\tau_3\big),\\ \gamma_3^2 &= r_3\big((I_2 + I_1)\tau_1 - \tau_2 - I_2I_1\tau_3\big), \end{aligned} \quad (28)$$

where

$$r_1 = \frac{(I_2 - I_3)I_1}{\Delta}, \qquad r_2 = \frac{(I_3 - I_1)I_2}{\Delta}, \qquad r_3 = \frac{(I_1 - I_2)I_3}{\Delta}.$$

Our last results on the Veselova problems are the following theorem and corollaries.

Theorem 3 *The following statements hold for the Veselova problem.*

(a) *System* (11) *for* $\Delta \neq 0$ *and* $\mu \neq 0$ (*i.e. we have a constrained system*) *admits the first integral*

$$\Phi_4 = \frac{1}{2}\|\gamma \wedge I\omega\|^2 - W(\tau_2, \tau_3) \quad (29)$$

if and only if the potential function U and the function W satisfy the partial differential equations of first order

$$\frac{\partial U}{\partial \tau_3} = \tilde{\nu}\tau_1, \qquad \frac{\partial W}{\partial \tau_2} = \tilde{\nu}\tau_3, \qquad |I|\frac{\partial U}{\partial \tau_2} + \frac{\partial W}{\partial \tau_3} = \big(-\tau_2 + (I_1 + I_2 + I_3)\tau_1\big)\tilde{\nu}, \quad (30)$$

where $|I| = I_1I_2I_3$ *and* (τ_1, τ_2, τ_3) *are the variables defined in* (27) *and* $\tilde{\nu} = \tilde{\nu}(\tau_2, \tau_3)$ *is an arbitrary function.*

(b) *System* (11) *with* $\Delta \neq 0$ *and* $\mu = 0$ (*i.e. the system has no constraints*) *admits the first integral* (29) *if and only if the potential function U and the function W satisfy the following partial differential equations of first order*:

$$\frac{\partial U}{\partial \tau_3} = 0, \qquad \frac{\partial W}{\partial \tau_2} = 0, \qquad |I|\frac{\partial U}{\partial \tau_2} + \frac{\partial W}{\partial \tau_3} = 0. \quad (31)$$

Clearly for the real motion $\tau_1 = 1$.

Corollary 2 *The following statements hold.*

(a) *System* (11) *with* $\Delta \neq 0$ *and* $\mu \neq 0$ *admits the first integral* Φ_4 *given by*

$$\frac{1}{2}\|\gamma \wedge I\omega\|^2 - \int_{\tau_2^0}^{\tau_2} \tau_3 \frac{\partial U}{\partial \tau_3}\bigg|_{\tau_3=\tau_3^0} d\tau_2 + \int_{\tau_3^0}^{\tau_3}\left(|I|\frac{\partial U}{\partial \tau_2} + \big(\tau_2 - (I_1 + I_2 + I_3)\big)\frac{\partial U}{\partial \tau_3}\right) d\tau_3 \quad (32)$$

if and only if the potential function U satisfies the linear second-order partial differential equations

$$2\frac{\partial U}{\partial \tau_3} + |I|\frac{\partial^2 U}{\partial \tau_2^2} + \big(\tau_2 - (I_1 + I_2 + I_3)\big)\frac{\partial^2 U}{\partial \tau_3\partial \tau_2} + \tau_3\frac{\partial^2 U}{\partial \tau_3^2} = 0. \quad (33)$$

(b) *System* (11) *with* $\Delta \neq 0$ *and* $\mu = 0$ *admits the first integral* Φ_4 *given by*

$$\frac{1}{2}\|\gamma \wedge I\omega\|^2 + \int_{\tau_3^0}^{\tau_3}\left(|I|\frac{\partial U}{\partial \tau_2}\right) d\tau_3 \quad (34)$$

if and only if the potential function U satisfies the linear second-order partial differential equations

$$\frac{\partial U}{\partial \tau_3} = 0, \qquad \frac{\partial^2 U}{\partial \tau_2\partial \tau_2} = 0.$$

Consequently, $U=\alpha\tau_2$, where α is a constant, is the Klebsh–Tisserand potential, and $W=-|I|\alpha\tau_3$. Thus, the first integral Φ_4 is

$$\frac{1}{2}\|\gamma\wedge I\omega\|^2+|I|\alpha\tau_3.$$

Corollary 3 *A particular solution of (33) is the potential function*

$$\begin{aligned}U&=a_0+a_1\tau_2+a_2\left(\tau_2^2-|I|\tau_3\right)\\&+\frac{\alpha_3}{r_1((I_2+I_3)\tau_1-\tau_2-I_2I_3\tau_3)}\\&+\frac{\alpha_4}{r_2((I_1+I_3)\tau_1-\tau_2-I_1I_3\tau_3)}\\&+\frac{\alpha_5}{r_3((I_2+I_1)\tau_1-\tau_2-I_2I_1\tau_3)},\end{aligned}\tag{35}$$

where a_0 and α_j for $j=3,4,5$ are constants. Consequently, we have the first integral

$$\begin{aligned}\Phi_4&=\frac{1}{2}\|\gamma\wedge I\omega\|^2-|I|a_2\left(\tau_2\tau_3+(I_1+I_2+I_3)\tau_3\right)\\&+|I|a_1\tau_3+\frac{\alpha_3(\tau_2-I_2-I_3)}{r_1((I_2+I_3)\tau_1-\tau_2-I_2I_3\tau_3)}\\&+\frac{\alpha_4(\tau_2-I_1-I_3)}{r_2((I_1+I_3)\tau_1-\tau_2-I_1I_3\tau_3)}\\&+\frac{\alpha_5(\tau_2-I_2-I_1)}{r_3((I_2+I_1)\tau_1-\tau_2-I_2I_1\tau_3)}.\end{aligned}$$

The proof of Theorem 3 and Corollaries 2 and 3 are given in Sect. 5.

Remark 4 Fedorov and Jovanovic in [6] claimed that

$$\begin{aligned}\Phi&=\frac{1}{2}\|\gamma\wedge I\omega\|^2-W(\tau_2,\tau_3)\\&=\frac{1}{2}\|\gamma\wedge I\omega\|^2+\alpha_1|I|\langle I\gamma,\gamma\rangle\langle I^{-1}\gamma,\gamma\rangle\\&-\alpha_2|I|\langle I^{-1}\gamma,\gamma\rangle+\alpha_3\left(I_2\frac{\gamma_2^2}{\gamma_1^2}+I_3\frac{\gamma_3^2}{\gamma_1^2}\right)\\&+\alpha_4\left(I_3\frac{\gamma_3^2}{\gamma_2^2}+I_1\frac{\gamma_1^2}{\gamma_2^2}\right)+\alpha_5\left(I_1\frac{\gamma_1^2}{\gamma_3^2}+I_2\frac{\gamma_2^2}{\gamma_3^2}\right)\end{aligned}\tag{36}$$

is a first integral of system (11) with the potential

$$\begin{aligned}U&=\alpha_1\left(\langle I^2\gamma,\gamma\rangle-\langle I\gamma,\gamma\rangle^2\right)+\alpha_2\langle I\gamma,\gamma\rangle\\&+\frac{\alpha_3}{\gamma_1^2}+\frac{\alpha_4}{\gamma_2^2}+\frac{\alpha_5}{\gamma_3^2},\end{aligned}\tag{37}$$

where α_j for $j=1,\dots,5$ are constants.

In fact, the first integral (36) of system (11) for the previous potential in the variables (τ_1,τ_2,τ_3) (see (28)) becomes

$$\begin{aligned}\Phi&=\frac{1}{2}\|\gamma\wedge I\omega\|^2-W(\tau_2,\tau_3)\\&=\frac{1}{2}\|\gamma\wedge I\omega\|^2+\alpha_1|I|\tau_2\tau_3-\alpha_2|I|\tau_3\\&\times\frac{\alpha_3\tau_2}{r_1((I_2+I_3)\tau_1-\tau_2-I_2I_3\tau_3)}\\&+\frac{\alpha_4\tau_2}{r_2((I_1+I_3)\tau_1-\tau_2-I_1I_3\tau_3)}\\&+\frac{\alpha_5\tau_2}{r_3((I_2+I_1)\tau_1-\tau_2-I_2I_1\tau_3)}\\&-\alpha_3I_1-\alpha_4I_2-\alpha_5I_3,\end{aligned}$$

The potential (37) and the function W defined in (36) write in the variables τ_2 and τ_3 as

$$\begin{aligned}U&=\alpha_1\left((I_1+I_2+I_3)\tau_2+|I|\tau_3\right.\\&\left.-(I_1I_2+I_2I_3+I_3I_1)-\tau_2^2\right)+\alpha_2\tau_2\\&+\frac{\alpha_3}{r_1((I_2+I_3)\tau_1-\tau_2-I_2I_3\tau_3)}\\&+\frac{\alpha_4}{r_2((I_1+I_3)\tau_1-\tau_2-I_1I_3\tau_3)}\\&+\frac{\alpha_5}{r_3((I_2+I_1)\tau_1-\tau_2-I_2I_1\tau_3)},\\W&=\alpha_1|I|\tau_2\tau_3-\alpha_2|I|\tau_3\\&\times\frac{\alpha_3\tau_2}{r_1((I_2+I_3)\tau_1-\tau_2-I_2I_3\tau_3)}\\&+\frac{\alpha_4\tau_2}{r_2((I_1+I_3)\tau_1-\tau_2-I_1I_3\tau_3)}\\&+\frac{\alpha_5\tau_2}{r_3((I_2+I_1)\tau_1-\tau_2-I_2I_1\tau_3)}\\&-\alpha_3I_1-\alpha_4I_2-\alpha_5I_3.\end{aligned}\tag{38}$$

Note that U coincide with the potential (35). Since W does not satisfy (30) with U given by (38), it follows that Φ is not a first integral.

3 Proof of Theorem 1

After some calculations, we obtain that the derivative of F_1 along the solutions of (9) takes the form

$$\begin{aligned}
\dot{F}_1 &= I_1\dot{\omega}_1 - \dot{\mu}_2 \\
&= \gamma_2\frac{\partial U}{\partial\gamma_3} - \gamma_3\frac{\partial U}{\partial\gamma_2} + \frac{\partial\mu_2}{\partial\gamma_1}\gamma_3\omega_2 - \frac{\partial\mu_2}{\partial\gamma_2}\gamma_3\omega_1 \\
&\quad - \frac{\partial\mu_2}{\partial\gamma_3}(\gamma_1\omega_2 - \gamma_2\omega_1) - \frac{\partial\mu_2}{\partial K_1}\dot{K}_1 - \frac{\partial\mu_2}{\partial K_2}\dot{K}_2 \\
&= \gamma_2\frac{\partial U}{\partial\gamma_3} - \gamma_3\frac{\partial U}{\partial\gamma_2} + \omega_2\left(-\gamma_1\frac{\partial\mu_2}{\partial\gamma_3} + \gamma_3\frac{\partial\mu_2}{\partial\gamma_1}\right) \\
&\quad + \omega_1\left(-\gamma_3\frac{\partial\mu_2}{\partial\gamma_2} + \gamma_2\frac{\partial\mu_2}{\partial\gamma_3}\right) \\
&= \gamma_2\frac{\partial U}{\partial\gamma_3} - \gamma_3\frac{\partial U}{\partial\gamma_2} + \frac{F_2-\mu_1}{I_2}\left(-\gamma_1\frac{\partial\mu_2}{\partial\gamma_3}\gamma_3\frac{\partial\mu_2}{\partial\gamma_1}\right) \\
&\quad + \frac{F_1+\mu_2}{I_1}\left(-\gamma_3\frac{\partial\mu_2}{\partial\gamma_2} + \gamma_2\frac{\partial\mu_2}{\partial\gamma_3}\right) \\
&= \gamma_2\frac{\partial}{\partial\gamma_3}\left(U + \frac{1}{2I_1I_2}(I_1\mu_1^2 + I_2\mu_2^2)\right) \\
&\quad - \gamma_3\frac{\partial}{\partial\gamma_2}\left(U + \frac{1}{2I_1I_2}(I_1\mu_1^2 + I_2\mu_2^2)\right) \\
&\quad + \frac{\mu_1}{I_2}\left(\gamma_3\left(\frac{\partial\mu_1}{\partial\gamma_2} - \frac{\partial\mu_2}{\partial\gamma_1}\right) - \gamma_2\frac{\partial\mu_1}{\partial\gamma_3} + \gamma_1\frac{\partial\mu_2}{\partial\gamma_3}\right) \\
&\quad - \frac{F_2}{I_2}\left(\gamma_1\frac{\partial\mu_2}{\partial\gamma_3} - \gamma_3\frac{\partial\mu_2}{\partial\gamma_1}\right) \\
&\quad - \frac{F_1}{I_1}\left(\gamma_3\frac{\partial\mu_2}{\partial\gamma_2} - \gamma_2\frac{\partial\mu_2}{\partial\gamma_3}\right),
\end{aligned}$$

here we use that $\dot{K}_1 = \dot{K}_2 = 0$.

In view of (14) and (15), we obtain

$$\begin{aligned}
\dot{F}_1 &= \frac{F_2}{I_2}\left(\gamma_1\frac{\partial\mu_2}{\partial\gamma_3} - \gamma_3\frac{\partial\mu_2}{\partial\gamma_1}\right) \\
&\quad + \frac{F_1}{I_1}\left(\gamma_3\frac{\partial\mu_2}{\partial\gamma_2} - \gamma_2\frac{\partial\mu_2}{\partial\gamma_3}\right).
\end{aligned}$$

A similar relation can obtained for $\dot{F}_2$. Hence, by considering (17), we deduce that $\dot{F}_j|_{F_1=F_2=0} = 0$ for $j = 1, 2$. Therefore, $F_1 = 0$ and $F_2 = 0$ are two general integrals. Consequently, there are two independent local first integrals

$$\Phi_1(\omega_1, \omega_2, \gamma_1, \gamma_2, \gamma_3) = K_3,$$

$$\Phi_4(\omega_1, \omega_2, \gamma_1, \gamma_2, \gamma_3) = K_4.$$

Thus, system (9) is locally explicitly integrable. We have a third general integral

$$F_3(\gamma_1, \gamma_2, \gamma_3, K_3)|_{K_3=1} = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 - 1 = 0.$$

On the other hand, system (9) has divergence zero due to the fact that its flow preserves the Lebesgue measure; see Proposition 1.

In short, by the Euler–Jacobi theorem it follows that system (9) is quasi-implicitly integrable. Hence, statement (a) is proved.

From Remark 2, first we consider the functions

$$\mu_1 = \frac{\partial\tilde{S}(\gamma_1, \gamma_2, \gamma_3, K_1, K_4)}{\partial\gamma_1} = \frac{\partial\tilde{S}}{\partial\gamma_1},$$

$$\mu_2 = \frac{\partial\tilde{S}(\gamma_1, \gamma_2, \gamma_3, K_1, K_4)}{\partial\gamma_2} = \frac{\partial\tilde{S}}{\partial\gamma_2}.$$

Hence, Eq. (17) become

$$\begin{aligned}
F_1 &= I_1\omega_1 - \frac{\partial\tilde{S}}{\partial\gamma_2} = 0, \\
F_2 &= I_2\omega_2 + \frac{\partial\tilde{S}}{\partial\gamma_1} = 0.
\end{aligned} \tag{39}$$

Now we show that the Suslov's, Kharlamova-Zabelina's, and Kozlov's first integral can be obtained from (39).

For the *Suslov integrable case* ($U =$ constant, and $I_1 \neq I_2$), we have that $\tilde{S} = C_1\gamma_1 + C_2\gamma_2$, where $C_1 = C_1(K_3, K_4)$ and $C_2 = C_2(K_3, K_4)$ are arbitrary constants. Since $\mu_1 = C_1$ and $\mu_2 = C_2$, taking

$$C_1 = \sqrt{\frac{I_2(I_1K_3 - K_4)}{I_1 - I_2}}, \qquad C_2 = \sqrt{\frac{I_1(K_4 - I_2K_3)}{I_1 - I_2}},$$

Eq. (16) holds. Equation (17) writes

$$F_1 = I_1\omega_1 - C_2 = 0, \qquad F_2 = I_2\omega_2 + C_1 = 0.$$

By solving this system with respect to K_3 and K_4, we obtain

$$K_3 = I_1\omega_1^2 + I_2\omega_2^2 = \frac{C_2^2}{I_2} + \frac{C_1^2}{I_1},$$

$$K_4 = I_1^2\omega_1^2 + I_2^2\omega_2^2 = C_2^2 + C_1^2.$$

Note that K_3 is the energy first integral, and K_4 is the kinetic moment (the Suslov first integral).

For the *Kharlamova–Zabelina integrable case* ($U = \langle \mathbf{b}, \gamma\rangle$), we take the function $\tilde{S}$ of the Remark 2 as

$$\tilde{S} = \frac{2/3}{\sqrt{I_1 b_1^2 + I_2 b_2^2}}(\tilde{h} + b_1\gamma_1 + b_2\gamma_2)^{3/2} - \frac{K_4}{b_1^2 I_1 + b_2^2 I_2}(b_2 I_2\gamma_1 - b_1 I_1\gamma_2),$$

where $\tilde{h} = I_1 I_2(\frac{K_4^2 I_1 I_2}{b_1^2 I_1 + b_2^2 I_2} - K_3)$, K_3 and K_4 are arbitrary constants. Then

$$\mu_1 = \frac{b_1}{\sqrt{I_1 b_1^2 + I_2 b_2^2}}\sqrt{\tilde{h} + b_1\gamma_1 + b_2\gamma_2} - \frac{K_4 b_2 I_2}{b_1^2 I_1 + b_2^2 I_2},$$

$$\mu_2 = \frac{b_2}{\sqrt{I_1 b_1^2 + I_2 b_2^2}}\sqrt{\tilde{h} + b_1\gamma_1 + b_2\gamma_2} + \frac{K_4 b_1 I_1}{b_1^2 I_1 + b_2^2 I_2}.$$

Therefore, Eq. (17) take the form

$$\begin{aligned} F_1 &= I_1\omega_1 - \left(\frac{b_2}{\sqrt{I_1 b_1^2 + I_2 b_2^2}}\sqrt{\tilde{h} + b_1\gamma_1 + b_2\gamma_2} + \frac{K_4 b_1 I_1}{b_1^2 I_1 + b_2^2 I_2}\right) = 0, \\ F_2 &= I_2\omega_2 + \left(\frac{b_1}{\sqrt{I_1 b_1^2 + I_2 b_2^2}}\sqrt{\tilde{h} + b_1\gamma_1 + b_2\gamma_2} - \frac{K_4 b_2 I_2}{b_1^2 I_1 + b_2^2 I_2}\right) = 0. \end{aligned} \tag{40}$$

Solving this system with respect to K_3 and K_4, we obtain

$$K_3 = I_1\omega_1^2 + I_2\omega_2^2 - \frac{1}{I_1 I_2}(b_1\gamma_1 + b_2\gamma_2),$$

$$K_4 = I_1\omega_1 b_1 + I_2\omega_2 b_2.$$

Again K_3 is the energy integral and K_4 is the well-known Kharlamova–Zabelina first integral [8].

For the *Kozlov's integrable case* ($U = a\gamma_3$ and $I_1 = I_2$), we take

$$\tilde{S} = -K_4 \arctan\left(\frac{\gamma_1}{\gamma_2}\right) + \frac{1}{2}\int D(\gamma_1^2 + \gamma_2^2)\, d(\gamma_1^2 + \gamma_2^2),$$

where

$$D(u) = I_1\sqrt{\frac{K_3 + a\sqrt{1-u}}{u} - \frac{K_4^2}{u^2}},$$

a, K_3 and K_4 are constants. Hence,

$$\mu_1 = -\frac{\gamma_2 K_4}{\gamma_1^2 + \gamma_2^2} + \gamma_1 D(\gamma_1^2 + \gamma_2^2),$$

$$\mu_2 = \frac{\gamma_1 K_4}{\gamma_1^2 + \gamma_2^2} + \gamma_2 D(\gamma_1^2 + \gamma_2^2).$$

Consequently, the functions F_1 and F_2 of (17) are

$$F_1 = \omega_1 - \left(\frac{\gamma_1 K_4}{\gamma_1^2 + \gamma_2^2} + \gamma_2 D(\gamma_1^2 + \gamma_2^2)\right) = 0,$$

$$F_2 = \omega_2 + \left(-\frac{\gamma_2 K_4}{\gamma_1^2 + \gamma_2^2} + \gamma_1 D(\gamma_1^2 + \gamma_2^2)\right) = 0.$$

Therefore, by solving this system with respect to K_3 and K_4, we obtain

$$K_3 = \omega_1^2 + \omega_2^2 - a\sqrt{1 - \gamma_1^2 - \gamma_2^2} = \omega_1^2 + \omega_2^2 - a\gamma_3,$$

$$K_4 = \omega_1\gamma_1 + \omega_2\gamma_2.$$

So, K_3 is the energy integral and K_4 is the Kozlov–Lagrange first integral. In fact, this integral correspond to the well-known integrable "Lagrange case" of the Suslov problem [10].

Finally, we analyze the case when the functions μ_1 and μ_2 of (19) are given by the formula

$$\begin{aligned} \mu_1 &= \Psi_1(\gamma_1^2 + \gamma_3^2, \gamma_1, K_3, K_4), \\ \mu_2 &= \Psi_2(\gamma_1^2 + \gamma_3^2, \gamma_2, K_3, K_4). \end{aligned} \tag{41}$$

We observe that these solutions, obtained integrating the system

$$\gamma_3\frac{\partial\mu_1}{\partial\gamma_2}-\gamma_2\frac{\partial\mu_1}{\partial\gamma_3}=0,\qquad \gamma_3\frac{\partial\mu_2}{\partial\gamma_1}-\gamma_1\frac{\partial\mu_2}{\partial\gamma_3}=0,$$

are a particular case of Eq. (15). The potential function (14) in this case coincides with the potential obtained by Dragović–Gajić–Jovanović in [4]. We call this case the *generalized Tisserand case*. In particular, if $I_1\neq I_2$ and

$$\mu_1=\sqrt{h_1+(a_1+a_3)\big(\gamma_3^2+\gamma_2^2\big)+(b_1+a_3)\gamma_1^2+f_1(\gamma_1)},$$

$$\mu_2=\sqrt{h_2+(a_2+a_4)\big(\gamma_3^2+\gamma_1^2\big)+(b_2+a_4)\gamma_2^2+f_2(\gamma_2)},$$

where $a_1, a_2, a_3, a_4, b_1, b_2$, and

$$h_1=\frac{I_2(I_1K_3-K_4)}{I_1-I_2},\qquad h_2=\frac{I_1(I_2K_3-K_4)}{I_1-I_2},$$

are constants, and f_1 and f_2 are arbitrary functions, then the general integrals $F_1=0$ and $F_2=0$ take the form

$$\begin{aligned}F_1&=I_1\omega_1\\&\quad-\sqrt{h_2+(a_2+a_4)\big(\gamma_3^2+\gamma_1^2\big)+(b_2+a_4)\gamma_2^2+f_2(\gamma_2)}\\&=0,\\F_2&=I_2\omega_2\\&\quad+\sqrt{h_1+(a_1+a_3)\big(\gamma_3^2+\gamma_2^2\big)+(b_1+a_3)\gamma_1^2+f_1(\gamma_1)}\\&=0.\end{aligned}$$

The case when $f_j(\gamma_j)=\alpha_j\gamma_j$, for $j=1,2$ was studied by Okuneva in [12], where α_1 and α_2 are constants.

If $f_1=f_2=0$, we obtain the Tisserand's case [9]. By solving $F_j=0$ for $j=1,2$ with respect to K_3 and K_4 we get that the first integrals in the Klebsh–Tisserand's case are

$$\begin{aligned}K_3&=I_1\omega_1^2+I_2\omega_2^2-\left(\frac{b_1+a_3}{I_2}+\frac{a_2+a_4}{I_1}\right)\gamma_1^2\\&\quad-\left(\frac{a_1+a_3}{I_2}+\frac{b_2+a_4}{I_1}\right)\gamma_2^2\\&\quad-\left(\frac{a_1+a_3}{I_2}+\frac{a_2+a_4}{I_1}\right)\gamma_3^2,\end{aligned}$$

$$\begin{aligned}K_4&=I_1^2\omega_1^2+I_2^2\omega_2^2-(b_1+a_3+a_2+a_4)\gamma_1^2\\&\quad-(a_1+a_3+b_2+a_4)\gamma_2^2\\&\quad-(a_1+a_3+a_2+a_4)\gamma_3^2.\end{aligned}$$

In short, statement (b) is proved.

Now we prove statement (c). First, we introduce the following well-known definition. An elliptic integral is any integral, which can be expressed in the form

$$\int R\big(x,\sqrt{P(x)}\big)\,dx,$$

where R is a rational function of its two arguments, P is a polynomial of degree 3 or 4 with no repeated roots. In general, elliptic integrals cannot be expressed in terms of elementary functions. Exceptions to this general rule are when P has repeated roots, or when $R(x,y)$ contains no odd powers of y. However, with the appropriate reduction formula, every elliptic integral can be brought into a form that involves integrals over rational functions and the three Legendre canonical forms (i.e., the elliptic integrals of the first, second, and third kind).

We observe that from the general integrals (17) it follows that

$$\omega_1=\frac{\mu_2}{I_1},\qquad \omega_2=-\frac{\mu_1}{I_2},$$

thus by inserting in the Poisson equation (see formula (9)) we easily obtain (18).

Now we deal with the case $\mu_j=\frac{\partial\tilde S(\gamma_1,\gamma_2,\gamma_3,K_3,K_4)}{\partial\gamma_j}$ for $j=1,2$. Therefore, Eq. (18) becomes

$$\begin{aligned}&\dot\gamma_1=-\frac{\gamma_3}{I_2}\frac{\partial\tilde S}{\partial\gamma_1},\qquad \dot\gamma_2=-\frac{\gamma_3}{I_1}\frac{\partial\tilde S}{\partial\gamma_2},\\&\dot\gamma_3=\frac{\gamma_1}{I_2}\frac{\partial\tilde S}{\partial\gamma_1}+\frac{\gamma_2}{I_1}\frac{\partial\tilde S}{\partial\gamma_2}.\end{aligned}\tag{42}$$

The vector $\gamma=\gamma(t)$ is determined integrating system (42).

For the Suslov case, the differential system (42) takes the form

$$\begin{aligned}&\dot\gamma_1=-\frac{\gamma_3C_1}{I_2},\qquad \dot\gamma_2=-\frac{\gamma_3C_2}{I_1},\\&\dot\gamma_3=\frac{\gamma_1C_1}{I_2}+\frac{\gamma_2C_2}{I_1}.\end{aligned}$$

After integration and taking into account that $\gamma_1^2+\gamma_2^2+\gamma_3^2=1$, we deduce that

$$\gamma_1(t)=\frac{C_1 I_1}{\sqrt{I_1^2C_1^2+I_2^2C_2^2}}\times\sin\beta\sin\left(\frac{\sqrt{I_1^2C_1^2+I_2^2C_2^2}}{I_1I_2}t+\alpha\right)+\frac{I_2C_2\cos\beta}{\sqrt{I_1^2C_1^2+I_2^2C_2^2}},$$

$$\gamma_2(t)=\frac{C_2 I_2}{\sqrt{I_1^2C_1^2+I_2^2C_2^2}}\times\sin\beta\sin\left(\frac{\sqrt{I_1^2C_1^2+I_2^2C_2^2}}{I_1I_2}t+\alpha\right)-\frac{I_1C_1\cos\beta}{\sqrt{I_1^2C_1^2+I_2^2C_2^2}},$$

$$\gamma_3(t)=\sin\beta\cos\left(\sqrt{\frac{I_1^2C_1^2+I_2^2C_2^2}{I_1I_2}}t+\alpha\right),$$

where α,β,C_1,C_2 are constants.

For the Kharlamova–Zabelina case, the differential system (18) is

$$\dot\gamma_1=\frac{\gamma_3}{I_2}\left(\frac{C_1}{\sqrt{I_1C_1^2+I_2C_2^2}}\sqrt{\tilde h+C_1\gamma_1+C_2\gamma_2}-\frac{K_4C_2I_2}{C_1^2I_1+C_2^2I_2}\right),$$

$$\dot\gamma_2=\frac{\gamma_3}{I_1}\left(\frac{C_2}{\sqrt{I_1C_1^2+I_2C_2^2}}\sqrt{\tilde h+C_1\gamma_1+C_2\gamma_2}+\frac{K_4C_1I_1}{C_1^2I_1+C_2^2I_2}\right),$$

$$\dot\gamma_3=-\frac{\gamma_1}{I_2}\left(\frac{C_1}{\sqrt{I_1C_1^2+I_2C_2^2}}\sqrt{\tilde h+C_1\gamma_1+C_2\gamma_2}-\frac{K_4C_2I_2}{C_1^2I_1+C_2^2I_2}\right)-\frac{\gamma_2}{I_1}\left(\frac{C_2}{\sqrt{I_1C_1^2+I_2C_2^2}}\sqrt{\tilde h+C_1\gamma_1+C_2\gamma_2}+\frac{K_4C_1I_1}{C_1^2I_1+C_2^2I_2}\right).$$

The solutions of this system are

$$\gamma_1=\frac{C_1}{I_2}(\tau-\tau_0)^2-\frac{K_4}{C_1^2I_1+C_2^2I_2}(\tau-\tau_1),$$

$$\gamma_2=\frac{C_2}{I_1}(\tau-\tau_0)^2-\frac{K_4}{C_1^2I_1+C_2^2I_2}(\tau-\tau_2),$$

$$\gamma_3=\sqrt{1-\gamma_1^2-\gamma_2^2}:=\sqrt{P_4(\tau,\tau_0,\tau_1,\tau_2,K_4)},$$

$$t=t_0+I_1I_2\int\frac{d\tau}{\sqrt{P_4(\tau,\tau_0,\tau_1,\tau_2,K_4)}},$$

where τ_0,τ_1,τ_2,K_4 are constants. Note that P_4 is a polynomial of degree four in the variable τ. Clearly, the equations of motion are integrable in elliptical functions of the time.

The differential system (42) in the Kozlov case takes the form

$$\dot\gamma_1=\gamma_3\left(\gamma_1D-\frac{\gamma_2-K_4}{1-\gamma_3^2}\right),$$

$$\dot\gamma_2=\gamma_3\left(\gamma_2D+\frac{\gamma_1-K_4}{1-\gamma_3^2}\right),\qquad \dot\gamma_3=-\left(1-\gamma_3^2\right)D,$$

where $D=D(1-\gamma_3^2)=D(\sin^2 z)$, and K_4 is an arbitrary constant. Hence, by considering the constraint $\omega_3=\dot x+\cos z\dot y=0$, we deduce the differential equations

$$\dot x=-\frac{K_4\cos z}{\sin^2 z},\qquad \dot y=\frac{K_4}{\sin^2 z},$$
$$\dot z=D\left(\sin^2 z\right)\sin z:=-\frac{\sqrt{P_3(\gamma_3,K_3,K_4,a)}}{\sin z},\tag{43}$$

which are easy to integrate. Note that P_3 is a polynomial of degree three in the variable γ_3. The solutions are

$$x=x_0-K_4\int\frac{\gamma_3dz}{\sin^3 zD(\sin^2 z)}=x_0+K_4\int\frac{\gamma_3d\gamma_3}{(1-\gamma_3^2)\sqrt{P_3(\gamma_3,K_3,K_4)}},$$

$$y=y_0+K_4\int\frac{dz}{\sin^3 zD(\sin^2 z)}=y_0-K_4\int\frac{d\gamma_3}{(1-\gamma_3^2)\sqrt{P_3(\gamma_3,K_3,K_4)}},$$

$$t=t_0+I_1I_2\int\frac{d\gamma_3}{\sqrt{P_3(\gamma_3,K_3,K_4)}}.$$

As we can observe in the Kozlov case as well as in Lagrange's classical problem of a heavy symmetric top x, y, and t are expressed as elliptic integral of γ_3.

For the generalized Tisserand's case, the dependence $\gamma = \gamma(t)$ is determined as follows. Let Γ_1 and Γ_2 be the following functions:

$$\Gamma_1 = \Gamma_1(\gamma_1) = \Psi_1(\gamma_2^2 + \gamma_3^2, \gamma_1)\big|_{\gamma_2^2+\gamma_3^2=1-\gamma_1^2},$$
$$\Gamma_2 = \Gamma_2(\gamma_2) = \Psi_2(\gamma_1^2 + \gamma_3^2, \gamma_2)\big|_{\gamma_1^2+\gamma_3^2=1-\gamma_2^2}.$$

Then the Poisson equations (18) take the form

$$\dot\gamma_1 = \frac{\gamma_3}{I_2}\Gamma_1, \qquad \dot\gamma_2 = \frac{\gamma_3}{I_1}\Gamma_2,$$
$$\dot\gamma_3 = -\frac{\gamma_1\Gamma_1}{I_2} - \frac{\gamma_2\Gamma_2}{I_2}. \tag{44}$$

The vector γ can be obtained as a function of time through the following quadratures:

$$\int\frac{d\gamma_1}{\Gamma_1(\gamma_1)} = I_1(\tau-\tau_0), \qquad \int\frac{d\gamma_2}{\Gamma_2(\gamma_2)} = I_2(\tau-\tau_0),$$
$$\sqrt{1-\gamma_1^2(\tau)-\gamma_2^2(\tau)} = \gamma_3,$$
$$\int\frac{d\tau}{\sqrt{1-\gamma_1^2(\tau)-\gamma_2^2(\tau)}} = \frac{t-t_0}{I_1I_2}.$$

For the Tisserand case, if we suppose that $h_1 + a_1 + a_3 > 0$, $a_1 - b_1 > 0$ and $h_2 + a_2 + a_4 > 0$, $a_2 - b_2 > 0$, after the integration of Eq. (44) we obtain

$$\gamma_1 = \sqrt{\frac{h_1+a_1+a_3}{a_1-b_1}}\sin\big(\sqrt{a_1-b_1}I_1(\tau-\tau_1)\big)$$
$$= \gamma_1(\tau),$$
$$\gamma_2 = \sqrt{\frac{h_2+a_2+a_4}{a_2-b_2}}\sin\big(\sqrt{a_2-b_2}I_2(\tau-\tau_2)\big)$$
$$= \gamma_2(\tau),$$
$$\gamma_3 = \sqrt{1-\gamma_1^2(\tau)-\gamma_2^2(\tau)},$$
$$t = t_0 + I_1I_2\int\frac{d\tau}{\sqrt{1-\gamma_1^2(\tau)-\gamma_2^2(\tau)}}.$$

If

$$\sqrt{a_1-b_1}I_1 = \sqrt{a_2-b_2}I_2 = \alpha, \qquad \tau_1=\tau_2=0,$$
$$\frac{h_1+h_2+a_1+a_2+a_3+a_4}{I_1-I_2} = k^2 > 0,$$

then

$$t = t_0 + I_1I_2\int\frac{d\tau}{\sqrt{1-k^2\sin^2(\alpha\tau)}}.$$

In the most general case, the analytical character of the solutions is essentially more complex. Hence, the proof of statement (c) follows. This completes the proof of Theorem 1.

Remark 5 (a) In all the known integrable cases studied before, our work condition (16) holds everywhere, and consequently we have local first integrals everywhere, but the known integrals are globally defined.

(b) In the previous results, we determine γ as a function of time through the quadratures. In general, these quadratures contain elliptic integrals, consequently to deduce the explicit form of the time dependence require previously to invert these integral. To obtain the time dependence of the angular velocity vector, we use the constraint $\omega_3 = 0$, and the general integrals $I_1\omega_1 - \mu_2 = 0$ and $I_2\omega_2 + \mu_1 = 0$.

4 Proof of Theorem 2 and Corollary 1

We start with some preliminary computations that are necessary for proving Theorem 2 and Corollary 3 (see [13]).

First, in view of (23), using the constraint $\gamma_1\omega_1 + \gamma_2\omega_2 + \gamma_3\omega_3 = \dot y + \dot x\cos z = 0$ and $\gamma_1^2+\gamma_2^2+\gamma_3^2 = 1$, we deduced the relations

$$\omega_1 = \frac{p\Psi_2\gamma_2 - I_2\Psi_1\gamma_1\gamma_3}{p(I_1\gamma_2^2 + I_2\gamma_1^2)},$$
$$\omega_2 = -\frac{p\Psi_2\gamma_1 + I_1\Psi_1\gamma_2\gamma_3}{p(I_1\gamma_2^2 + I_2\gamma_1^2)}, \qquad \omega_3 = \frac{\Psi_1}{p}. \tag{45}$$

Consequently,

$$\langle I\omega, \omega\rangle = \frac{\Psi_1^2 + \Psi_2^2}{I_1\gamma_2^2 + I_2\gamma_1^2}. \tag{46}$$

From (45), the Poisson equations become (25), which in view of (4) we get

$$\dot{x} = \frac{\Psi_1}{\sin^2 zp}, \qquad \dot{y} = -\frac{\cos z \Psi_1}{\sin^2 zp},$$
$$\dot{z} = \frac{\Psi_2 p + (I_1 - I_2)\Psi_1 \cos x \sin x \cos z}{p \sin z (I_1 \cos^2 x + I_2 \sin^2 x)}. \tag{47}$$

From the Poisson equations (25) and Eq. (47), we obtain the relation

$$\begin{aligned}
&\frac{dG(\gamma)}{dt} \\
&= (\gamma_2\omega_3 - \gamma_3\omega_2)\frac{\partial G}{\partial \gamma_1} \\
&\quad + (\gamma_3\omega_1 - \gamma_1\omega_3)\frac{\partial G}{\partial \gamma_2} + (\gamma_1\omega_2 - \gamma_2\omega_1)\frac{\partial G}{\partial \gamma_3} \\
&= \frac{((I_2 - I_1)\langle \gamma, \frac{\partial G}{\partial \gamma}\rangle \gamma_1\gamma_2 + I_1\gamma_2\frac{\partial G}{\partial \gamma_1} - I_2\gamma_1\frac{\partial G}{\partial \gamma_2})\Psi_1}{p(I_1\gamma_2^2 + I_2\gamma_1^2)} \\
&\quad + \frac{p\Psi_2(\gamma_3\langle \gamma, \frac{\partial G}{\partial \gamma}\rangle - \frac{\partial G}{\partial \gamma_3})}{p(I_1\gamma_2^2 + I_2\gamma_1^2)} \\
&= \frac{\Psi_1}{p(I_1\gamma_2^2 + I_2\gamma_1^2)}\Big(\big(I_1\cos^2 x + I_2 \sin^2 x\big)\frac{\partial G}{\partial x} \\
&\quad + (I_1 - I_2)\cos x \sin x \cos z \sin z \frac{\partial G}{\partial z}\Big) \\
&\quad + \frac{p\Psi_2 \sin z}{p(I_1\gamma_2^2 + I_2\gamma_1^2)}\frac{\partial G}{\partial z}.
\end{aligned}$$

Thus, using that Ψ_1 and Ψ_2 are arbitrary functions, we get that

$$\begin{aligned}
&(I_2 - I_1)\Big\langle \gamma, \frac{\partial G}{\partial \gamma}\Big\rangle \gamma_1\gamma_2 + I_1\gamma_2\frac{\partial G}{\partial \gamma_1} - I_2\gamma_1\frac{\partial G}{\partial \gamma_2} \\
&\quad = \big(I_1\cos^2 x + I_2\sin^2 x\big)\frac{\partial G}{\partial x} \\
&\qquad + (I_1 - I_2)\cos x \sin x \cos z \sin z \frac{\partial G}{\partial z}, \\
&\gamma_3\Big\langle \gamma, \frac{\partial G}{\partial \gamma}\Big\rangle - \frac{\partial G}{\partial \gamma_3} = \sin z \frac{\partial G}{\partial z}.
\end{aligned} \tag{48}$$

Now we calculate the derivative of $\Psi_1 = \Psi_j(x, z)$ for $j = 1, 2$ along the solutions of (47) and using (21), we obtain

$$\begin{aligned}
\frac{d\Psi_2}{dt} &= \frac{\Psi_1}{p(I_1\gamma_2^2 + I_2\gamma_1^2)}\big(I_1\cos^2 x + I_2\sin^2 x\big)\frac{\partial \Psi_2}{\partial x} \\
&\quad + \frac{1}{(I_1\gamma_2^2 + I_2\gamma_1^2)}(I_1 - I_2) \\
&\quad \times \cos x \sin x \cos z \sin z \frac{\partial \Psi_2}{\partial z} \\
&\quad + \frac{\sin z}{2(I_1\gamma_2^2 + I_2\gamma_1^2)}\frac{\partial \Psi_2^2}{\partial z} \\
&= \Theta + \frac{\sin z}{(I_1\gamma_2^2 + I_2\gamma_1^2)}\frac{\partial \Psi}{\partial z} \\
&= \Theta + \sin z \frac{\partial}{\partial z}\left(\frac{\Psi}{I_1\gamma_2^2 + I_2\gamma_1^2}\right) \\
&\quad + 2\cos z \frac{\Psi}{I_1\gamma_2^2 + I_2\gamma_1^2}, \\
\frac{d\Psi_1}{dt} &= \frac{\Psi_1}{p(I_1\gamma_2^2 + I_2\gamma_1^2)}\Big(\big(I_1\cos^2 x + I_2\sin^2 x\big)\frac{\partial \Psi_1}{\partial x} \\
&\quad + (I_1 - I_2)\cos x \sin x \cos z \sin z \frac{\partial \Psi_1}{\partial z}\Big) \\
&\quad + \frac{p\Psi_2 \sin z}{p(I_1\gamma_2^2 + I_2\gamma_1^2)}\frac{\partial \Psi_1}{\partial z} \\
&= \Theta + \frac{1}{p(I_1\gamma_2^2 + I_2\gamma_1^2)} \\
&\quad \times \big(I_1\cos^2 x + I_2\sin^2 x\big)\frac{\partial \Psi}{\partial x} \\
&\quad + \frac{1}{p(I_1\gamma_2^2 + I_2\gamma_1^2)} \\
&\quad \times (I_1 - I_2)\cos x \sin x \cos z \sin z \frac{\partial \Psi}{\partial z} \\
&= \Theta + \frac{1}{(I_1\gamma_2^2 + I_2\gamma_1^2)}\Big((I_2 - I_1)\Big\langle \gamma, \frac{\partial \Psi}{\partial \gamma}\Big\rangle \gamma_1\gamma_2 \\
&\quad + I_1\gamma_2\frac{\partial \Psi}{\partial \gamma_1} - I_2\gamma_1\frac{\partial \Psi}{\partial \gamma_2}\Big) \\
&= \Theta + (I_2 - I_1)\gamma_1\gamma_2\left(\left\langle \gamma, \frac{\partial}{\partial \gamma}\left(\frac{\Psi}{I_1\gamma_2^2 + I_2\gamma_1^2}\right)\right\rangle\right. \\
&\quad \left. + 2\frac{\Psi}{I_1\gamma_2^2 + I_2\gamma_1^2}\right) + I_1\gamma_2\frac{\partial}{\partial \gamma_1}\left(\frac{\Psi}{I_1\gamma_2^2 + I_2\gamma_1^2}\right) \\
&\quad - I_2\gamma_1\frac{\partial}{\partial \gamma_2}\left(\frac{\Psi}{I_1\gamma_2^2 + I_2\gamma_1^2}\right),
\end{aligned} \tag{49}$$

where $\Psi = \frac{1}{2}(\Psi_1^2 + \Psi_3^2)$.

Proof of Theorem 2 After some tedious computations from (11), (45), (46), (47), (48), and (49), we deduce the relations

$$\frac{dF_1}{dt} = \frac{d}{dt}(I_1\omega_1\gamma_2 - I_2\omega_2\gamma_1) - \frac{d}{dt}\Psi_2$$
$$= -\Theta - \sin z\frac{\partial}{\partial z}\left(U + \frac{\Psi}{I_1\gamma_2^2 + I_2\gamma_1^2}\right)$$
$$+ \cos z\left(\langle I\omega, \omega\rangle - \frac{2\Psi}{I_1\gamma_2^2 + I_2\gamma_1^2}\right).$$

Thus, $\frac{dF_1}{dt}|_{F_1=F_2=0} = -\Theta$. Moreover,

$$\frac{dF_2}{dt} = \frac{d(p\omega)}{dt} - \frac{d\Psi_1}{dt}$$
$$= -\Theta - (I_2 - I_1)\gamma_1\gamma_2$$
$$\times\left(\left\langle\gamma, \frac{\partial}{\partial\gamma}\left(U + \frac{\Psi}{I_1\gamma_2^2 + I_2\gamma_1^2}\right)\right\rangle\right.$$
$$\left. + \langle I\omega, \omega\rangle - \frac{2\Psi}{I_1\gamma_2^2 + I_2\gamma_1^2}\right)$$
$$+ I_1\gamma_2\frac{\partial}{\partial\gamma_1}\left(U + \frac{\Psi}{I_1\gamma_2^2 + I_2\gamma_1^2}\right)$$
$$- I_2\gamma_1\frac{\partial}{\partial\gamma_2}\left(U + \frac{\Psi}{I_1\gamma_2^2 + I_2\gamma_1^2}\right).$$

Hence, $\frac{dF_2}{dt}|_{F_1=F_2=0} = -\Theta$.

Here, we apply the relation

$$\langle I\omega, \omega\rangle = 2(h - U)$$
$$= \frac{\Psi_1^2 + \Psi_2^2}{I_1\gamma_2^2 + I_2\gamma_1^2} \quad \text{with } h = \text{constant},$$

obtained from energy integral in view of (46), and

$$\frac{d(p\omega)}{dt} = (I_1 - I_2)\gamma_1\gamma_2\left(\left\langle\gamma, \frac{\partial U}{\partial\gamma}\right\rangle + \langle I\omega, \omega\rangle\right)$$
$$+ I_1\gamma_2\frac{\partial U}{\partial\gamma_1} - I_2\gamma_1\frac{\partial U}{\partial\gamma_2},$$

$$\frac{d}{dt}(I_1\omega_1\gamma_2 - I_2\omega_2\gamma_1)$$
$$= -\gamma_3\left\langle\gamma, \frac{\partial U}{\partial\gamma}\right\rangle + \frac{\partial U}{\partial\gamma_3} + \gamma_3\langle I\omega, \omega\rangle,$$

which we deduce from (12) and (13). Consequently, in view of (21), we have $\frac{dF_j}{dt}|_{F_1=F_2=0} = 0$, for $j = 1, 2$. Hence, $F_1 = 0$ and $F_2 = 0$ are general integrals. This proves statement (a) of the theorem.

To prove statement (b), first we observe that if $\Psi_1^2 + \Psi_2^2 = 2\Psi(x)$, then from (49) we obtain that $\frac{d\Psi_2}{dt} = \Theta$. Thus, if (21) holds then $\Psi_2 = K_4 =$ arbitrary constant. Consequently, from statement (a), we obtain the first integral $I_1\omega_1\gamma_2 - I_2\omega_2\gamma_1 = K_4$. By considering the energy integral, and in view of the Euler–Jacobi theorem we obtain the proof of statement (b).

Now we prove statement (c). If $I_1 \neq I_2$ and $\Psi_1^2 + \Psi_2^2 = 2\Psi(\frac{I_1\gamma_2^2+I_2\gamma_1^2}{\gamma_3^2})$, then from (49) it follows the proof of this statement by considering that the function $\Psi(\frac{I_1\gamma_2^2+I_2\gamma_1^2}{\gamma_3^2})$ is a solution of the equation

$$0 = (I_2 - I_1)\left\langle\gamma, \frac{\partial\Psi}{\partial\gamma}\right\rangle\gamma_1\gamma_2 + I_1\gamma_2\frac{\partial\Psi}{\partial\gamma_1} - I_2\gamma_1\frac{\partial\Psi}{\partial\gamma_2}$$
$$= \left(I_1\cos^2 x + I_2\sin^2 x\right)\frac{\partial\Psi}{\partial x}$$
$$+ (I_1 - I_2)\cos x\sin x\cos z\sin z\frac{\partial\Psi}{\partial z}.$$

For proving statement (d), first we observe that the partial differential equation (21) for $I_1 = I_2$ takes the form

$$\frac{\partial\Psi_2}{\partial x} - \sin z\sqrt{\alpha + (1-\alpha)\cos^2 z}\frac{\partial\Psi_1}{\partial z} = 0, \quad \alpha = \frac{I_3}{I_1}. \tag{50}$$

Hence,

$$\frac{\partial}{\partial x}\left(\frac{\Psi_2}{\sin z\sqrt{\alpha + (1-\alpha)\cos^2 z}}\right) - \frac{\partial\Psi_1}{\partial z} = 0.$$

Thus, we get that

$$\Psi_2 = \sin z\sqrt{\alpha + (1-\alpha)\cos^2 z}\frac{\partial S}{\partial z}, \qquad \Psi_1 = \frac{\partial S}{\partial x}. \tag{51}$$

where $S = S(x, z)$ is an arbitrary function. If $\Psi_1^2 + \Psi_2^2 = 2\Psi(z, K_3, K_4)$, then from (21) we deduce that $\frac{d\Psi_1}{dt} = \Theta$, consequently in view of (21) we obtain that $\Psi_1 = K_4 =$ arbitrary constant.

In particular if $S = S_1(z) + K_4 x$, then $\Psi_1 = K_4$, and consequently from (51) we obtain the first integral $\sqrt{\alpha + (1-\alpha)\cos^2 z}\,\omega_3 = K_4$. By considering the energy integral, and in view of the Euler–Jacobi theorem we obtain the proof of statement (c).

Finally, we observe that differential system (47) when $\Psi_1 = K_4$ and $I_1 = I_2$ takes the form

$$\dot{x} = \frac{K_4}{\sin^2 z\sqrt{\alpha + (1-\alpha)\cos^2 z}},$$
$$\dot{y} = \frac{-\cos z K_4}{\sin^2 z\sqrt{\alpha + (1-\alpha)\cos^2 z}},$$
$$\dot{z} = \frac{\sqrt{\alpha + (1-\alpha)\cos^2 z}}{I_1} S_1'.$$

The solutions of these equations are easy to obtain because it is of separable variables. This completes the proof of Theorem 2. □

Proof of Corollary 3 The proof follows from the relations (48). □

5 Proof of Theorem 1 and Corollary 2

Proof of Theorem 1 From (12) and after an easy computation, we have

$$\frac{d}{dt}\left(\|\gamma \wedge I\omega\|^2\right) = -2\left\langle \frac{\partial U}{\partial \gamma}, \gamma \wedge I\omega \right\rangle.$$

The function $\frac{1}{2}\|\gamma \wedge I\omega\|^2 - W(\gamma_1,\gamma_2,\gamma_3)$ is a first integral if and only if

$$\left\langle \frac{\partial U}{\partial \gamma}, \gamma \wedge I\omega \right\rangle = \left\langle \frac{\partial W}{\partial \gamma}, \gamma \wedge \omega \right\rangle,$$

consequently

$$\left\langle I\left(\frac{\partial U}{\partial \gamma} \wedge \gamma\right) - \frac{\partial W}{\partial \gamma} \wedge \gamma, \omega \right\rangle = 0.$$

Thus, by considering that $\langle \gamma, \omega \rangle = 0$, we have

$$I\left(\frac{\partial U}{\partial \gamma} \wedge \gamma\right) - \frac{\partial W}{\partial \gamma} \wedge \gamma = \lambda \gamma, \tag{52}$$

or equivalently

$$\frac{\partial U}{\partial \gamma} \wedge \gamma - I^{-1}\left(\frac{\partial W}{\partial \gamma} \wedge \gamma\right) = \lambda I^{-1}\gamma,$$

where $\lambda = \lambda(\gamma_1,\gamma_2,\gamma_3)$ is an arbitrary function.

To solve the partial differential equations (52) when $\Delta \neq 0$, we use the relations

$$\frac{\partial f}{\partial \gamma_j} = 2\gamma_j\left(\frac{\partial \bar{f}}{\partial \tau_1} + I_j\frac{\partial \bar{f}}{\partial \tau_2} + \frac{1}{I_j}\frac{\partial \bar{f}}{\partial \tau_3}\right),$$

for $j = 1,2,3$, where $f = f(\gamma_1,\gamma_2,\gamma_3) = \bar{f}(\tau_1,\tau_2,\tau_3)$. Hence,

$$\frac{\partial f}{\partial \gamma} \wedge \gamma = \upsilon\left(\frac{(I_2-I_3)}{\gamma_1}\left(\frac{\partial \bar{f}}{\partial \tau_2} - \frac{I_1}{|I|}\frac{\partial \bar{f}}{\partial \tau_3}\right), \frac{(I_3-I_1)}{\gamma_2}\left(\frac{\partial \bar{f}}{\partial \tau_2} - \frac{I_2}{|I|}\frac{\partial f}{\partial \tau_3}\right), \frac{(I_1-I_2)}{\gamma_3}\left(\frac{\partial \bar{f}}{\partial \tau_2} - \frac{I_3}{|I|}\frac{\partial \bar{f}}{\partial \tau_3}\right)\right),$$

where $\upsilon = 2\gamma_1\gamma_2\gamma_3$. After some computations, we obtain

$$I\left(\frac{\partial U}{\partial \gamma} \wedge \gamma\right) - \frac{\partial W}{\partial \gamma} \wedge \gamma = \upsilon\left(\frac{(I_2-I_3)}{\gamma_1}\Phi_1, \frac{(I_3-I_1)}{\gamma_2}\Phi_2, \frac{(I_1-I_2)}{\gamma_3}\Phi_3\right) = \lambda\gamma,$$

$$\Phi_j = I_j\frac{\partial U}{\partial \tau_2} - \frac{I_j^2}{|I|}\frac{\partial U}{\partial \tau_3} - \frac{\partial W}{\partial \tau_2} + \frac{I_j}{|I|}\frac{\partial W}{\partial \tau_3},$$

for $j = 1,2,3$. Thus,

$$(I_2-I_3)\Phi_1 = \frac{\lambda}{2\gamma_1\gamma_2\gamma_3}\gamma_1^2,$$
$$(I_3-I_1)\Phi_2 = \frac{\lambda}{2\gamma_1\gamma_2\gamma_3}\gamma_2^2,$$
$$(I_1-I_2)\Phi_3 = \frac{\lambda}{2\gamma_1\gamma_2\gamma_3}\gamma_3^2.$$

Hence, by using the relations

$$\frac{I_2-I_3}{I_1} + \frac{I_3-I_1}{I_2} + \frac{I_1-I_2}{I_3} = \frac{I_1^2(I_3-I_2) + I_2^2(I_1-I_3) + I_3^2(I_2-I_1)}{I_1I_2I_3} = \frac{(I_1-I_2)(I_2-I_3)(I_3-I_1)}{I_1I_2I_3} \neq 0,$$

$$I_1^3(I_2-I_3) + I_2^3(I_3-I_1) + I_3^3(I_1-I_2) = -(I_1-I_2)(I_2-I_3)(I_3-I_1)(I_1+I_2+I_3),$$

we deduce that

$$\begin{aligned}
&\frac{(I_1-I_2)(I_2-I_3)(I_3-I_1)}{I_1I_2I_3}\frac{\partial U}{\partial\tau_3}=\frac{\lambda}{\gamma_1\gamma_2\gamma_3}\tau_1,\\
&\frac{(I_1-I_2)(I_2-I_3)(I_3-I_1)}{I_1I_2I_3}\frac{\partial W}{\partial\tau_2}=\frac{\lambda}{\gamma_1\gamma_2\gamma_3}\tau_3,\\
&\frac{(I_1-I_2)(I_2-I_3)(I_3-I_1)}{I_1I_2I_3}\\
&\quad\times\left(I_1I_2I_3\frac{\partial U}{\partial\tau_2}+\frac{\partial W}{\partial\tau_3}-(I_1+I_2+I_3)\frac{\partial U}{\partial\tau_3}\right)\\
&\quad=-\frac{\lambda\tau_2}{\gamma_1\gamma_2\gamma_3}.
\end{aligned}\tag{53}$$

Denoting by $\tilde{\nu}=\frac{I_1I_2I_3\lambda}{(I_1-I_2)(I_2-I_3)(I_3-I_1)\gamma_1\gamma_2\gamma_3}$ from (53), we easily obtain (30). This is the proof of statement (a).

Now we consider the case $\Delta\neq 0$ and $\mu=0$. The equations of motion for a rigid body without constraints are the Euler–Poisson equations

$$I\dot{\omega}=I\omega\wedge\omega+\gamma\wedge\frac{\partial U}{\partial\gamma},\qquad \dot{\gamma}=\gamma\wedge\omega.$$

The necessary and sufficient conditions for the existence of a first integral K_4 are obtained from (52) with $\lambda=0$, because the system is free of constraints. Hence, from (53), we get the condition (31). This completes the proof of the theorem. □

Proof of Corollary 2 From (30) and considering that $\tau_1=1$, we have

$$\begin{aligned}
&\frac{\partial W}{\partial\tau_2}=\tau_3\frac{\partial U}{\partial\tau_3},\\
&\frac{\partial W}{\partial\tau_3}=-|I|\frac{\partial U}{\partial\tau_2}+\big(-\tau_2+(I_1+I_2+I_3)\big)\frac{\partial U}{\partial\tau_3}.
\end{aligned}\tag{54}$$

The compatibility condition of this system is given by Eq. (33). Hence, if U satisfies Eq. (33), then the function W is obtained integrating (54). Consequently, from (29) the require first integral takes the form (32). In short, statement (a) is proved.

Under the assumptions of statement (b), the potential function U must satisfy Eq. (31). So $\tilde{\nu}=0$. Now the proof of statement (b) is easy to obtain from the proof of the previous statement. Thus, the theorem is proved. □

Proof of Corollary 3 The potential function

$$U=a_0+a_1\tau_2+a_2\big(\tau_2^2-|I|\tau_3\big)+\frac{a_4}{a_7+a_5\tau_2+a_6\tau_3},$$

where

$$a_6=\frac{a_4(|I|a_4-a_5(I_1+I_2+I_3)}{a_5},\tag{55}$$

for arbitrary $a_0,a_1,a_2,a_3,a_4,a_5\neq 0$, is a particular solution of (33). Then the first integral (32) in this case becomes

$$\begin{aligned}
\Phi_4={}&\frac{1}{2}\|\gamma\wedge I\omega\|^2-|I|\alpha\big(\tau_2\tau_3+(I_1+I_2+I_3)\tau_3\big)\\
&+|I|\beta\tau_3+\frac{a_3(a_4\tau_2+a_6)}{a_6+a_4\tau_2+a_5\tau_3}.
\end{aligned}$$

Consequently, from (54), we get that

$$\begin{aligned}
W={}&\frac{a_4(a_5\tau_2+a_7)}{a_7+a_5\tau_2+a_6\tau_3}\\
&+|I|a_2\big(\tau_2\tau_3+(I_1+I_2+I_3\tau_3)\big)-|I|a_1\tau_3.
\end{aligned}$$

We denote by $w=a_7+a_5\tau_2+a_6\tau_3$. We study the following particular cases:

$$\begin{aligned}
&a_4=\frac{\alpha_3}{r_1},\qquad a_5=r_1,\qquad a_6=r_1I_1I_2,\\
&a_7=-r_1(I_3+I_2)\quad\Longrightarrow\quad w=\gamma_1^2,\\
&a_4=\frac{\alpha_4}{r_2},\qquad a_5=r_2,\qquad a_6=r_2I_1I_3,\\
&a_7=-r_2(I_3+I_1)\quad\Longrightarrow\quad w=\gamma_2^2,\\
&a_4=\frac{\alpha_5}{r_3},\qquad a_5=r_3,\qquad a_6=r_3I_1I_3,\\
&a_7=-r_3(I_3+I_1)\quad\Longrightarrow\quad w=\gamma_3^2,
\end{aligned}$$

where r_1,r_2,r_3 are constants given in (28). By considering that (54) and (33) are linear partial differential equations, we obtain that the linear combination of the functions

$$\begin{aligned}
&U_1=\frac{1}{\gamma_1^2},\qquad U_2=\frac{1}{\gamma_2^2},\qquad U_3=\frac{1}{\gamma_3^2},\\
&W_1=\frac{I_2(1-\gamma_2^2)+I_3(1-\gamma_3^2)+I_1\gamma_1^2)}{\gamma_1^2},\\
&W_2=\frac{I_1(1-\gamma_1^2)+I_3(1-\gamma_3^2)+I_2\gamma_2^2)}{\gamma_2^2},
\end{aligned}$$

$$W_3 = \frac{I_2(1-\gamma_2^2)+I_1(1-\gamma_1^2)+I_3\gamma_3^2)}{\gamma_3^2},$$

are solutions of these equations. Therefore, for the potential function (35) or what is the same

$$U = a_0 + a_1\tau_2 + a_2(\tau_2^2 - |I|\tau_3) + \alpha_3 U_1 + \alpha_4 U_2 + \alpha_5 U_3,$$
$$W = |I|a_2(\tau_2\tau_3 + (I_1+I_2+I_3\tau_3)) + |I|a_1\tau_3 + \alpha_4 W_1 + \alpha_5 W_2 + \alpha_6 W_3$$

we obtain the first integral

$$\begin{aligned}\Phi_4 = {} & \frac{1}{2}\|\gamma\wedge I\omega\|^2 - |I|a_2(\tau_2\tau_3 + (I_1+I_2+I_3\tau_3)) \\ & + |I|a_1\tau_3 \\ & + \frac{\alpha_3(I_2(1-\gamma_2^2)+I_3(1-\gamma_3^2)+I_1\gamma_1^2)}{\gamma_1^2} \\ & + \frac{\alpha_4((I_1(1-\gamma_1^2)+I_3(1-\gamma_3^2)+I_2\gamma_2^2)}{\gamma_2^2} \\ & + \frac{\alpha_5((I_2(1-\gamma_2^2)+I_1(1-\gamma_1^2)+I_3\gamma_3^2))}{\gamma_3^2},\end{aligned} \tag{56}$$

where α_j for $j=3,4,5$ are constants. □

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