Laplacian eigenvalues and partition problems in hypergraphs*

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Abstract

We use the generalization of the Laplacian matrix to hypergraphs to obtain several spectral-like results on partition problems in hypergraphs which are computationally difficult to solve (NP-hard or NP-complete). Therefore it is very important to obtain nontrivial bounds. More precisely, the following parameters are bounded in the paper: bipartition width, averaged minimal cut, isoperimetric number, max-cut, independence number and domination number.

Keywords: Laplacian Matrix; Hypergraph; Bipartition width; Averaged minimal cut; Isoperimetric number; Max-cut; Independence number; Domination number.

AMS Subject Classification numbers: 05C50; 05A20; 05C65; 05C35

1 Laplacian Matrix

Throughout this paper $\mathcal{H} = (V, E)$ denotes a simple and finite hypergraph with vertex set $V = V(\mathcal{H})$, $|V| = n$, and edge set $E = E(\mathcal{H})$, $|E| = m$. The *degree* $\delta(v)$ of the vertex v is defined as the cardinality of the edge set containing v. A hypergraph in which all vertices have the same degree, δ, is called δ*-regular*, and if all edges have the same cardinality r, it is called r*-uniform*. The class of r-uniform hypergraphs contains, for instance, the class of graphs ($r = 2$) and the class of block designs. Throughout this paper $\Gamma = (V, E)$ denotes a graph.

We denote by $\mathbf{A} = \mathbf{A}(\mathcal{H})$ the adjacency matrix of $\mathcal{H}$. Given two distinct vertices $v_i, v_j \in V(\mathcal{H})$ the entry a_{ij} of $\mathbf{A}$ is the number of edges in $\mathcal{H}$ containing both v_i and v_j; the diagonal entries of $\mathbf{A}$ are zero. We define the *Laplacian degree of a vertex* $v_i \in V(\mathcal{H})$ as $\delta_\ell(v_i) = \sum_{j=1}^{n} a_{ij}$. We say that the hypergraph $\mathcal{H}$ is *Laplacian regular* of degree δ_ℓ if any vertex $v \in V(\mathcal{H})$ has Laplacian degree $\delta_\ell(v) = \delta_\ell$. If $\mathcal{H}$ is a graph then $\delta_\ell(v_i) = \delta(v_i)$.

The *Laplacian matrix of a hypergraph* $\mathcal{H}$, denoted by $\mathbf{L} = \mathbf{L}(\mathcal{H})$, is defined as $\mathbf{L} := \mathcal{D} - \mathbf{A}$ where $\mathcal{D} = diag(\delta_\ell(v_1), \delta_\ell(v_2), \ldots, \delta_\ell(v_n))$. This version of Laplacian matrix was introduced by the author in [12] and [13] to extend, to the case of hypergraphs, results related with several

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metric parameters of graphs. Here we will continue extending the graph eigenvalue results to hypergraphs focusing our study in the relation between the second smallest and largest Laplacian eigenvalues and parameters related to partition problems.

We recall that the matrix $\mathbf{L}$ is symmetric and positive semidefinite, the smallest eigenvalue of $\mathbf{L}$ is $\mu = 0$ and a corresponding eigenvector is $\mathbf{j} = (1, 1, \ldots, 1)$. Moreover, the multiplicity of $\mu = 0$ is equal to the number of connected components of $\mathcal{H}$.

The eigenvalues of $\mathbf{L}$ are denoted by $\mu_0 = 0 < \mu_1 < \cdots < \mu_b$ and their multiplicities are denoted by $m_0 = 1, m_1, \ldots, m_b$. Thus, the Laplacian spectrum of $\mathcal{H}$ and the Laplacian degrees of its vertices are related by the following equality

$$\sum_{l=1}^{b} m_l \mu_l = Tr(\mathbf{L}(\mathcal{H})) = \sum_{i=1}^{n} \delta_\ell(v_i).$$

Let $\Gamma = (V, \hat{E})$ be the weighted graph on the same vertex set V as $\mathcal{H}$, in which two vertices $v_i, v_j \in V(\Gamma)$ are adjacent if they are adjacent in $\mathcal{H}$, and the edge-weight of the edge $v_i v_j$ is equal to (a_{ij}) the number of edges in $\mathcal{H}$ containing both v_i and v_j. Clearly, $\mathbf{L}(\mathcal{H}) = \mathbf{L}(\Gamma)$. Thus, the second smallest Laplacian eigenvalue of $\mathcal{H}$, μ_1, satisfies the following equality showed by Fiedler [2] on weighted graphs

$$\mu_1 = 2n \min \left\{ \frac{\sum_{v_i \sim v_j} a_{ij}(w_i - w_j)^2}{\sum_{v_i \in V} \sum_{v_j \in V} (w_i - w_j)^2} : w \neq \alpha \mathbf{j} \text{ for } \alpha \in \mathbb{R} \right\}, \tag{1}$$

It is well-known that the second smallest Laplacian eigenvalue of a graph is probably the most important information contained in the spectrum. This eigenvalue, frequently called *algebraic connectivity*, is related to several important graph invariants and imposes reasonably good bounds on the values of several parameters of graphs which are very hard to compute. In this paper we use (1) to obtain nontrivial bounds of several parameters related to partition problems in hypergraphs. In all of them μ_1 can be viewed as measures of connectivity.

Similarly to (1), the largest Laplacian eigenvalue, μ_b, satisfies

$$\mu_b = 2n \max \left\{ \frac{\sum_{v_i \sim v_j} a_{ij}(w_i - w_j)^2}{\sum_{v_i \in V} \sum_{v_j \in V} (w_i - w_j)^2} : w \neq \alpha \mathbf{j} \text{ for } \alpha \in \mathbb{R} \right\}, \tag{2}$$

and (2) will become our second more important tool of the paper.

We identify the Laplacian matrix $\mathbf{L}$ with an endomorphism of the "vertex-space" of $\mathcal{H}$, $l^2(V(\mathcal{H}))$ which, for any given indexing of the vertices, is isomorphic to $\mathbb{R}^n$. Thus, for any vertex $v_i \in V(\mathcal{H})$, e_i will denote the corresponding unit vector of the canonical base of $\mathbb{R}^n$.

By putting $w = e_i$ in (1) and (2) we get $\mu_1 \leq \frac{n}{n-1}\delta_\ell(v_i) \leq \mu_b$. Since e_i was chosen arbitrarily, the above inequalities lead to

$$\mu_1 \leq \frac{n}{n-1}\delta_{\ell \min} \leq \frac{n}{n-1}\delta_{\ell \max} \leq \mu_b. \tag{3}$$

We denote by $\lambda_0 > \lambda_1 > \cdots > \lambda_d$ the adjacency eigenvalues of a Laplacian regular hypergraph $\mathcal{H}$ of degree δ_ℓ. Then, since $\mathbf{L} = \delta_\ell \mathbf{I} - \mathbf{A}$, the eigenvalues of both matrices, $\mathbf{A}$ and $\mathbf{L}$, are related by $\mu_l = \delta_\ell - \lambda_l, \quad l = 0, \ldots, b = d$. Notice also that δ_ℓ is the trivial eigenvalue of $\mathbf{A}$ with $\mathbf{j}$ as eigenvector. Hence, in this case, the matrices $\mathbf{A}$ and $\mathbf{L}$ lead to equivalent spectral-like results.

2 Coboundary

The *edge cut* or *coboundary*, E_X, of the set $X \subset V(\mathcal{H})$ is defined as the set of all edges $\varepsilon \in E(\mathcal{H})$ such that there are two vertices $u, v \in \varepsilon$ with $u \in X$ and $v \notin X$. The minimum cardinality of E_X, over all nontrivial subsets X of $V(\mathcal{H})$, is called edge connectivity of $\mathcal{H}$, and the maximum is the cardinality version of the max-cut of $\mathcal{H}$ denoted by $mc(\mathcal{H})$. Obviously, the subsets $X \subset V(\mathcal{H})$ and $Y = V(\mathcal{H})\backslash X$ have the same coboundary: $E_X = E_Y$. Using (1) and (2) we obtain the following bounds on $|E_X|$.

Lemma 1. *Let $\mathcal{H}$ be a simple r-uniform hypergraph of order n, and let μ_1 and μ_b be its second smallest and largest Laplacian eigenvalues. If $X \subset V(\mathcal{H})$, then*

$$\frac{\mu_b |X| (n - |X|)}{(r-1)n} \geq |E_X| \geq \begin{cases} \frac{4\mu_1 |X|(n-|X|)}{r^2 n} & \text{if } r \text{ is even;} \\ \frac{4\mu_1 |X|(n-|X|)}{(r^2-1)n} & \text{if } r \text{ is odd.} \end{cases} \tag{4}$$

Proof. Obviously, the bound holds for $X = \emptyset$ and $X = V(\mathcal{H})$. Thus, we consider $\emptyset \neq X \subset V(\mathcal{H})$. Let $w = \sum_{v_i \in X} e_i$ be the vector associated to X. Then,

$$\sum_{v_i \in V(\mathcal{H})} \sum_{v_j \in V(\mathcal{H})} (w_i - w_j)^2 = 2 |X| (n - |X|). \tag{5}$$

By (1), (2) and (5) we have

$$\mu_1 \leq \frac{n \sum_{v_i \sim v_j} a_{ij}(w_i - w_j)^2}{|X| (n - |X|)} \leq \mu_b. \tag{6}$$

On the other hand, considering the 2-partition $\{X, V(\mathcal{H})\backslash X\}$ of the vertex set $V(\mathcal{H})$ we have $(w_i - w_j)^2 = 1$ if v_i and v_j are in different sets of the partition, and 0 if they are in the same set. Thus, if v_i and v_j are adjacent and they are in different sets of the partition, $a_{ij}(w_i - w_j)^2$ is the number of edges of $\mathcal{H}$ containing both v_i and v_j. Moreover, if $\varepsilon \in E(\mathcal{H})$ and $|\varepsilon \cap X| = k$ $(1 \leq k < r)$, then $\sum_{v_i \sim v_j} (w_i - w_j)^2$ count $k(r-k)$ times the edge ε. The maximum value of the discrete function $f(k) = k(r-k)$ is attained on $k = \left[\frac{r}{2}\right]$; $k = \frac{r}{2}$ if r is even and $k = \frac{r-1}{2}$ if r is odd. Therefore,

$$|E_X| \geq \frac{\sum_{v_i \sim v_j} a_{ij}(w_i - w_j)^2}{f(\frac{r}{2})} = \frac{4 \sum_{v_i \sim v_j} a_{ij}(w_i - w_j)^2}{r^2}$$

if r is even, and

$$|E_X| \geq \frac{\sum_{v_i \sim v_j} a_{ij}(w_i - w_j)^2}{f(\frac{r-1}{2})} = \frac{4 \sum_{v_i \sim v_j} a_{ij}(w_i - w_j)^2}{(r^2 - 1)}$$

if r is odd. By the left side of (6) and the above inequalities we conclude the proof of the lower bounds.

Moreover, the minimum value of the discrete function $f(k) = k(r-k)$, $1 \leq k \leq r-1$, is attained on $k = 1$ and $k = r-1$. Then,

$$|E_X| \leq \frac{\sum_{v_i \sim v_j} a_{ij}(w_i - w_j)^2}{f(1)} = \frac{\sum_{v_i \sim v_j} a_{ij}(w_i - w_j)^2}{r-1}.$$

By the right side of (6) and the above inequality we conclude the proof of the upper bound. □

In the case of graphs we have $r = 2$, then the above lemma generalizes the previous results on graphs (see, for instance [8]). If $X \subset V(\Gamma)$, then

$$\frac{\mu_1 |X| (n - |X|)}{n} \leq |E_X| \leq \frac{\mu_b |X| (n - |X|)}{n}. \tag{7}$$

Some of the partition problems discussed in this paper are essentially related to finding an appropriate subset $X \subset V(\mathcal{H})$ such that the coboundary E_X satisfies some extremal property. Computationally such problems are NP-hard. Therefore it is very important to obtain nontrivial bounds. Concretely, Lemma 1 leads to spectral-like bounds on the bipartition width, the averaged minimal cut, the isoperimetric number and the max-cut.

3 Bisection

A *bisection* of $\mathcal{H}$ is a 2-partition$\{X, Y\}$ of the vertex set $V(\mathcal{H})$ in which $|X| = |Y|$ or $|X| = |Y| + 1$. The bisection problem is to find a bisection for which $|E_X|$ is as small as possible. The *bipartition width* $bw(\mathcal{H})$ of the hypergraph $\mathcal{H}$ is defined as

$$bw(\mathcal{H}) := \min \left\{ |E_X| : X \subset V(\mathcal{H}), |X| = \left\lfloor \frac{n}{2} \right\rfloor \right\}.$$

Theorem 2. *Let μ_1 be the second smallest Laplacian eigenvalue of a simple r-uniform hypergraph $\mathcal{H}$ of order n. Then,*

$$bw(\mathcal{H}) \geq \begin{cases} \left\lceil \frac{n\mu_1}{r^2} \right\rceil & \textit{if } r \textit{ and } n \textit{ are even;} \\ \left\lceil \frac{n\mu_1}{r^2-1} \right\rceil & \textit{if } r \textit{ is odd and } n \textit{ is even;} \\ \left\lceil \frac{(n^2-1)\mu_1}{nr^2} \right\rceil & \textit{if } r \textit{ is even and } n \textit{ is odd;} \\ \left\lceil \frac{(n^2-1)\mu_1}{n(r^2-1)} \right\rceil & \textit{if } r \textit{ and } n \textit{ are odd.} \end{cases}$$

Proof. Taking $|X| = \frac{n}{2}$ if n is even and $|X| = \frac{n-1}{2}$ if n is odd in the right inequalities of Lemma 1, the result follows. $\square$

The above bound is tight as we can see in the following example. Let $\mathcal{H}$ be the hypergraph defined by the vertex set $V(\mathcal{H}) = \{1, 2, 3, 4, 5, 6, 7, 8\}$, and the edge set $E(\mathcal{H}) = \{\{1, 2, 3, 4\}, \{3, 4, 5, 6\}, \{5, 6, 7, 8\}, \{7, 8, 1, 2\}\}$ (see Figure 1). In this case $\mu_1 = 4$ from which Theorem 2 gives the sharp bound $|E_X| \geq \frac{n\mu_1}{r^2} = 2$.

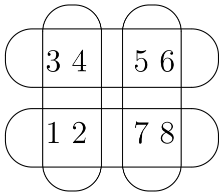
The above theorem is a generalization of a previous result on graphs that motivated this study (see, for instance, Merris [7] and Mohar [8, 9]):

$$bw(\Gamma) \geq \begin{cases} \left\lceil \frac{n\mu_1}{4} \right\rceil & \textit{if } n \textit{ is even;} \\ \left\lceil \frac{(n^2-1)\mu_1}{4n} \right\rceil & \textit{if } n \textit{ is odd.} \end{cases}$$

4 Averaged minimal cut

We define the *edge-density* of vertex set $X \subset V(\mathcal{H})$ as $\rho_c(X) := \frac{|E_X|}{|X|(n-|X|)}$, and it represents the density of the edges between the set X and its complement.

Figure 1: The bound is tight



Theorem 3. *Let μ_1 and μ_b be the second smallest and largest Laplacian eigenvalues of a simple r-uniform hypergraph $\mathcal{H}$ of order n. For any nontrivial subset X of vertices of $\mathcal{H}$,*

$$\frac{\mu_b}{(r-1)n} \geq \rho_c(X) \geq \begin{cases} \frac{4\mu_1}{r^2 n} & \text{if } r \text{ is even;} \\ \\ \frac{4\mu_1}{(r^2-1)n} & \text{if } r \text{ is odd.} \end{cases}$$

Proof. By definition of $\rho_c(X)$ and Lemma 1, the result immediately follow. □

The *averaged minimal cut* of $\mathcal{H}$ is defined as $\gamma(\mathcal{H}) := \min_{0<|X|<n} \rho_c(X)$. Evidently, the above result imposes a nontrivial lower bound on $\gamma(\mathcal{H})$. An example in which the bound is attained is the hypergraph of Figure 1. Taking $X = \{1,2,3,4\}$ we have $|E_X| = 2$. Moreover, in this case, $n = 8$, $\mu_1 = 4$ and $r = 4$ from which the above bound gives $\gamma(\mathcal{H}) \geq \frac{1}{8}$.

In the case of graphs the result is simplified as the following well-known bound (see [10])

$$\gamma(\Gamma) \geq \frac{\mu_1}{n}. \tag{8}$$

5 Isoperimetric number

We define the *isoperimetric number* of a hypergraph $\mathcal{H}$ of order n as the quantity

$$i(\mathcal{H}) := \min\left\{\frac{|E_X|}{|X|} : \emptyset \neq X \subset V(\mathcal{H}), |X| \leq \frac{n}{2}\right\}$$

The isoperimetric number of a graph has been extensively studied, for instance, we cite the papers by Mohar [8, 9], Kahale [4] and Kwak *et al.* [5]. Here we study the case of r-uniform hypergraphs.

Theorem 4. *Let μ_1 be the second smallest Laplacian eigenvalue of a simple r-uniform hypergraph $\mathcal{H}$. Then,*

$$i(\mathcal{H}) \geq \begin{cases} \frac{2\mu_1}{r^2} & \text{if } r \text{ is even;} \\ \\ \frac{2\mu_1}{r^2-1} & \text{if } r \text{ is odd.} \end{cases}$$

Proof. By Lemma 1 we have

$$\frac{|E_X|}{|X|} \geq \begin{cases} \frac{4\mu_1(n-|X|)}{r^2 n}, & \text{if r is even;} \\ \\ \frac{4\mu_1(n-|X|)}{(r^2-1)n}, & \text{if r is odd.} \end{cases} \tag{9}$$

Moreover, if $|X| \leq \frac{n}{2}$, then $\frac{n-|X|}{n} \geq \frac{1}{2}$. Hence, the result follows. □

The above bound is tight. For instance, let $\mathcal{H}$ be the hypergraph of Figure 1, then $i(\mathcal{H}) \geq \frac{1}{2}$. In this case $r = 4$ and $\mu_1 = 4$ from which Theorem 4 gives the sharp bound $i(\mathcal{H}) \geq \frac{1}{2}$.

A particular case of the above bound, when $\mathcal{H} = \Gamma$ is a graph, is the Mohar's bound [9]:

$$i(\Gamma) \geq \frac{\mu_1}{2} \tag{10}$$

6 Max-cut

The maximum cardinality of E_X, over all nontrivial subsets X of $V(\mathcal{H})$, is called the cardinality version of the *max-cut* of $\mathcal{H}$ denoted by $mc(\mathcal{H})$. That is,

$$mc(\mathcal{H}) := \max \{|E_X| : \emptyset \neq X \subset V(\mathcal{H})\}.$$

Lemma 1 leads to the following upper bound on $mc(\mathcal{H})$.

Theorem 5. *Let μ_b be the largest Laplacian eigenvalue of a simple r-uniform hypergraph $\mathcal{H}$ of order n. Then,*

$$mc(\mathcal{H}) \leq \begin{cases} \left\lfloor \frac{n\mu_b}{4(r-1)} \right\rfloor & \text{if } n \text{ is even;} \\ \left\lfloor \frac{(n^2-1)\mu_b}{4n(r-1)} \right\rfloor & \text{if } n \text{ is odd.} \end{cases}$$

Proof. The maximum value of the discrete function $f(x) = x(n - x)$ is attained on $x = \left[\frac{n}{2}\right]$; $n = \frac{n}{2}$ if n is even and $x = \frac{n-1}{2}$ if n is odd. Therefore, Lemma 1 leads to the result. □

The above bound generalizes, to the case of hypergraphs, the previous one given by Mohar and Poljak on graphs [11]:

$$mc(\Gamma) \leq \begin{cases} \left\lfloor \frac{n\mu_b}{4} \right\rfloor & \text{if } n \text{ is even;} \\ \left\lfloor \frac{(n^2-1)\mu_b}{4n} \right\rfloor & \text{if } n \text{ is odd.} \end{cases}$$

7 Independence number

An independent set of $\mathcal{H}$ is a set $X \subset V(\mathcal{H})$ such that none of the neighbors of a vertex $v \in X$ is in X. The *independence number* $\alpha(\mathcal{H})$ of $\mathcal{H}$ is the cardinality of a largest independent set of $\mathcal{H}$.

As the independent set problem is NP-complete [3], the following nontrivial bound take singular importance.

Theorem 6. *Let μ_b be the largest Laplacian eigenvalue of a simple r-uniform hypergraph $\mathcal{H}$ of order n. Let $\delta_{\min}$ be the minimum degree of $\mathcal{H}$. Then, $\alpha(\mathcal{H}) \leq \left\lfloor \frac{n(\mu_b - \delta_{\min}(r-1))}{\mu_b} \right\rfloor$.*

Proof. Let X be an independent set of $\mathcal{H}$. Then, the coboundary of X is bounded below as $|E_X| \geq \delta_{\min} |X|$ Moreover, if $w = \sum_{v_i \in X} e_i$, then $|E_X| = \frac{\sum_{v_i \sim v_j} a_{ij}(w_i - w_j)^2}{r-1}$. Thus, by (6), the result follows. □

For instance, if $\mathcal{H}$ is the hypergraph of Figure 1 in which $\mu_b = 8$ and $\delta_{\min} = 2$, the above result gives the sharp bound $\alpha(\mathcal{H}) \leq 2$. In particular, if $\mathcal{H} = \Gamma$ is a graph, we obtain the following result: $\alpha(\Gamma) \leq \left\lfloor \frac{n(\mu_b - \delta_{\min})}{\mu_b} \right\rfloor$. For instance, in the case of the t-dimensional hypercubes, $\mu_b = 2t$ and $n = 2^t$ from which the above result gives the sharp bound $\alpha(\Gamma) \leq 2^{t-1}$. Another example is the Petersen graph $\mathcal{P}$ in which $n = 10$, $\mu_b = 5$ and $\delta_{\min} = 3$, hence the above bound gives again the sharp bound $\alpha(\mathcal{P}) \leq 4$. In fact, if Γ is regular, the above bound coincides with the Hoffman-Lovász' bound [1, 6].

8 Domination number

A *dominating set* of a graph Γ is a set $X \subset V(\Gamma)$ such that every vertex $v \in V(\Gamma) \setminus X$ has at least one neighbor in X. The *domination number* $\Upsilon(\Gamma)$ of Γ is the cardinality of a smallest dominating set of Γ. The dominating set problem is also NP-complete [3].

If X is a dominating set of a δ-regular graph Γ, then $\delta\,|X| \geq n - |X|$. Thus, we have $\Upsilon(\Gamma) \geq \left\lceil \frac{n}{\delta+1} \right\rceil$. This bound is attained, for example, in the case of $\Gamma = K_m \times K_2$. In this case, $n = 2m$ and $\delta = m$ from which the bound gives $\Upsilon(\Gamma) \geq \left\lceil \frac{2m}{m+1} \right\rceil = 2$. Another example in which the bound is attained is the case of the 4-cube. In this case $n = 16$ and $\delta = 4$ from which we have $\Upsilon(\Gamma) \geq \left\lceil \frac{16}{5} \right\rceil = 4$.

We generalize the concept of domination to hypergraphs. A *k-dominating set* of a hypergraph $\mathcal{H}$ is a set $X \subset V(\mathcal{H})$ such that every vertex $v \in V(\mathcal{H}) \setminus X$ belong to at least k edges $\varepsilon_1, ..., \varepsilon_k \in E(\mathcal{H})$ where $\varepsilon_i \cap X \neq \emptyset$, $i = 1, ..., k$. The *k-domination number* $\Upsilon_k(\mathcal{H})$ is the cardinality of a smallest k-dominating set of $\mathcal{H}$.

Theorem 7. *Let μ_b be the largest Laplacian eigenvalue of a simple uniform hypergraph $\mathcal{H}$ of order n. Then, $\Upsilon_k(\mathcal{H}) \geq \left\lceil \frac{nk}{\mu_b} \right\rceil$.*

Proof. Let $\mathcal{H}$ be an r-uniform hypergraph and let X be a k-dominating set of $\mathcal{H}$. If $w = \sum_{v_i \in X} e_i$, then $\frac{\sum_{v_i \sim v_j} a_{ij}(w_i - w_j)^2}{r-1} \geq |E_X| \geq \frac{k(n-|X|)}{r-1}$. Moreover, by (2) we have $\frac{|X|(n-|X|)\mu_b}{n} \geq \sum_{v_i \sim v_j} a_{ij}(w_i - w_j)^2$. Hence, the result follows. □

The above bound is tight as we can see in the following examples. Let $\mathcal{H}$ be the hypergraph whose edges are $b_1 = \{1, 2, 3\}$, $b_2 = \{2, 3, 4\}$, $b_3 = \{3, 4, 5\}$, $b_4 = \{4, 5, 6\}$, $b_5 = \{5, 6, 7\}$, $b_6 = \{6, 7, 8\}$, $b_7 = \{7, 8, 9\}$, $b_8 = \{8, 9, 10\}$, $b_9 = \{9, 10, 11\}$, $b_{10} = \{10, 11, 12\}$, $b_{11} = \{11, 12, 1\}$ and $b_{12} = \{12, 1, 2\}$. In this case a 3-dominating set of $\mathcal{H}$ is $X = \{1, 4, 7, 10\}$. As $\mu_b = 9$, Theorem 7 gives $\Upsilon_3(\mathcal{H}) \geq 4$.

To show the tightness of the above bound in the case of graphs we take the infinity family of t-dimensional hypercubes. In this case $\mu_b = 2t$ and $n = 2^t$ from which Theorem 7 also gives the good bound $\Upsilon_t(\Gamma) \geq 2^{t-1}$.

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