Bounds on the Number of Numerical Semigroups of a Given Genus

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February 18, 2008

Abstract

Combinatorics on multisets is used to deduce new upper and lower bounds on the number of numerical semigroups of each given genus, significantly improving existing ones. In particular, it is proved that the number n_g of numerical semigroups of genus g satisfies $2F_g \leqslant n_g \leqslant 1 + 3 \cdot 2^{g-3}$, where F_g denotes the gth Fibonacci number.

1 Introduction

Let $\mathbb{N}_0$ denote the set of all non-negative integers. A *numerical semigroup* is a subset Λ of $\mathbb{N}_0$ containing 0, closed under summation and with finite complement in $\mathbb{N}_0$. The elements in the complement $\mathbb{N}_0 \setminus \Lambda$ are called the *gaps* of the numerical semigroup and $|\mathbb{N}_0 \setminus \Lambda|$ is its *genus*. The largest gap is the *Frobenius number* of Λ and it is at most two times the genus minus one. If it equals this bound then the numerical semigroup is said to be symmetric.

Some results have been proved related to the number of numerical semigroups of a given Frobenius number [1] and the number of symmetric semigroups of a given Frobenius number (and thus, the number of symmetric semigroups of a given genus) [4, 8]. In this work we address the problem of counting the number of numerical semigroups of a given genus.

We denote by n_g the number of numerical semigroups of genus g. It is easy to check that $n_0 = 1$ and $n_1 = 1$. The values up to n_{16} where computed by Nivaldo Medeiros and Shizuo Kakutani, and the values up to n_{50} can be found in [2]. It is proved in [3] that any numerical semigroup can be represented by a unique Dyck path of order given by its genus and thus $n_g \leqslant C_g$ where C_g denotes the Catalan number, $C_g = \frac{1}{g+1}\binom{2g}{g}$. It is conjectured in [2] that the sequence (n_g) asymptotically behaves like the Fibonacci sequence. More precisely, $n_g \geqslant n_{g-1}+n_{g-2}$, for $g \geqslant 2$; $\lim_{g\to\infty} \frac{n_{g-1}+n_{g-2}}{n_g} = 1$; $\lim_{g\to\infty} \frac{n_g}{n_{g-1}} = \phi$, where ϕ is the golden ratio. See [5] for further results in this direction.

*This work was partly supported by the Spanish Ministry of Education through projects TSI2007-65406-C03-01 "E-AEGIS" and CONSOLIDER CSD2007-00004 "ARES", and by the Government of Catalonia under grant 2005 SGR 00446.

Let F_i denote the ith Fibonacci number starting by $F_0 = 0$, $F_1 = 1$. We prove that

$$2F_g \leqslant n_g \leqslant 1 + 3 \cdot 2^{g-3}.$$

2 Some Results on Combinatorics

Lemma 1. *The multisets A_g defined recursively by $A_2 = \{1, 3\}$,*

$$A_g = \{g+1\} \cup \left(\bigcup_{m \in A_{g-1}} \{0, 1, \ldots, m-1\} \right) \setminus \{g-2\}$$

for $g > 2$ (see Figure 1) satisfy, if $g \geqslant 2$,

$$A_g = \left(\overbrace{\{0,0,\ldots,0\}}^{2F_{g-2}} \cup \overbrace{\{1,1,\ldots,1\}}^{2F_{g-3}} \cup \overbrace{\{2,2,\ldots,2\}}^{2F_{g-4}} \cup \ldots \cup \overbrace{\{g-4,g-4\}}^{2F_2} \cup \overbrace{\{g-3,g-3\}}^{2F_1} \right) \cup \{g-1, g+1\}$$

and

$$|A_g| = 2F_g.$$

Proof. Both results can be proved by induction and are a consequence from the fact that, for $i \geqslant 2$, $F_i = 1 + \sum_{j=1}^{i-2} F_j$. This in turn can be proved by induction. Indeed, it is obvious for $i = 2$. If $i > 2$, by the induction hypothesis $F_{i-1} = 1 + \sum_{j=1}^{i-3} F_j$ and hence $F_i = F_{i-1} + F_{i-2} = 1 + \sum_{j=1}^{i-2} F_j$. □

$A_2 = \{1, 3\}$
$A_3 = \{0, 0, 2, 4\}$
$A_4 = \{0, 0, 1, 1, 3, 5\}$
$A_5 = \{0, 0, 0, 0, 1, 1, 2, 2, 4, 6\}$
$A_6 = \{0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 2, 2, 3, 3, 5, 7\}$
$A_7 = \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 4, 4, 6, 8\}$

Figure 1: First multisets A_g as in Lemma 1.

Lemma 2. *The multisets B_g defined recursively by $B_2 = \{1, 3\}$,*

$$B_g = \{0, g+1\} \cup \left(\bigcup_{m \in B_{i-1}} \{1, 2, \ldots, m\} \right) \setminus \{g, g-2\}$$

for $g > 2$ (see Figure 2) satisfy, if $g > 2$,

$$B_g = \{0\} \cup \left(\{\overbrace{1,1,\dots,1}^{3\cdot 2^{g-4}}\} \cup \{\overbrace{2,2,\dots,2}^{3\cdot 2^{g-5}}\} \cup \dots \cup \{\overbrace{g-3,g-3,g-3}^{3\cdot 2^{0}}\} \right) \cup \{g-2, g-1, g+1\}$$

and

$$|B_g| = 1 + 3 \cdot 2^{g-3}.$$

Proof. Both results can be proved by induction and are a consequence from the fact that, for $i \geqslant 0$, $2^i = 1 + \sum_{j=0}^{i-1} 2^j$. This in turn can be proved by induction. □

$B_2 = \{1, 3\}$
$B_3 = \{0, 1, 2, 4\}$
$B_4 = \{0, 1, 1, 1, 2, 3, 5\}$
$B_5 = \{0, 1, 1, 1, 1, 1, 1, 2, 2, 2, 3, 4, 6\}$
$B_6 = \{0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 3, 3, 3, 4, 5, 7\}$
$B_7 = \{0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,$
$2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 4, 4, 4, 5, 6, 8\}$

Figure 2: First multisets B_g as in Lemma 2.

3 Taking out Generators from a Semigroup

Every numerical semigroup can be generated by a finite set of elements and a minimal set of generators is unique (see for instance [4]). Given a numerical semigroup Λ of genus g and Frobenius number f, $\Lambda \cup \{f\}$ is a numerical semigroup and its genus is $g-1$. So, any numerical semigroup of genus g can be obtained from a numerical semigroup of genus $g-1$ by removing one element larger than its Frobenius number. It is easy to check that when removing such an element from a numerical semigroup, the set obtained is a numerical semigroup if and only if the removed element belongs to the set of minimal generators. This gives a recursive procedure to obtain all numerical semigroups of genus g from all numerical semigroups of genus $g-1$ by taking out, one by one, each generator that is larger than the Frobenius number for each numerical semigroup.

We can think of a tree whose root corresponds to the numerical semigroup $\mathbb{N}_0$, each numerical semigroup of genus g is a node at distance g from the root, and the children of a numerical semigroup are the numerical semigroups obtained when removing one by one each of its minimal generators which are larger than its Frobenius number. This construction was already considered in [6, 8, 7]. We depicted this tree in Figure 3. We wrote $< \lambda_{i_1}, \lambda_{i_2}, \dots, \lambda_{i_n} >$ to denote the numerical semigroup generated by $\lambda_{i_1}, \lambda_{i_2}, \dots, \lambda_{i_n}$. We used boldface letters for the minimal generators that are larger than the Frobenius number.

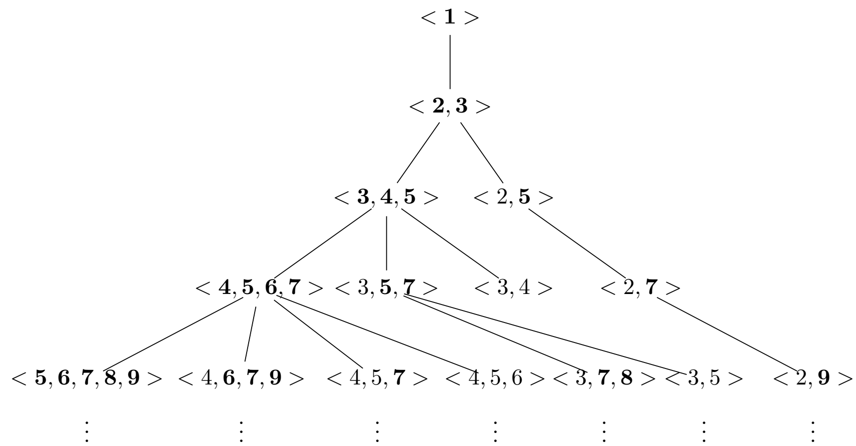

Figure 3: Recursive construction of numerical semigroups of genus g from numerical semigroups of genus $g-1$.

We say that a numerical semigroup is *ordinary* if it is equal to $\{0\} \cup \{i \in \mathbb{N}_0 : i \geqslant c\}$ for some non-negative integer c.

Lemma 3. *Let Λ be a non-ordinary numerical semigroup. Suppose that $\{\lambda_{i_1} < \lambda_{i_2} < \ldots < \lambda_{i_k}\}$ are the minimal generators of Λ which are larger than the Frobenius number. Then the number of minimal generators of the numerical semigroup $\Lambda \setminus \{\lambda_{i_j}\}$ which are larger than its Frobenius number is*

- *at least $k-j$,*
- *at most $k-j+1$.*

Proof. It is obvious that the number of minimal generators which are larger than the Frobenius number is at least $k-j$ because all elements in $\Lambda \setminus \{\lambda_{i_j}\}$ which are minimal generators in Λ are also minimal generators in $\Lambda \setminus \{\lambda_{i_j}\}$ and the new Frobenius number is λ_{i_j}.

The elements in $\Lambda \setminus \{\lambda_{i_j}\}$ which are not minimal generators in Λ and become minimal generators in $\Lambda \setminus \{\lambda_{i_j}\}$ must be of the form $\lambda_{i_j} + \lambda_r$ for some $\lambda_r \in \Lambda$. Let λ_1 be the smallest non-zero element of Λ. If $\lambda_r > \lambda_1$ then $\lambda_{i_j} + \lambda_r - \lambda_1 > \lambda_{i_j}$ and hence $\lambda_{i_j} + \lambda_r = \lambda_1 + \lambda_s$ for some $\lambda_s \in \Lambda \setminus \{\lambda_{i_j}\}$, and $\lambda_{i_j} + \lambda_r$ is not a minimal generator of $\Lambda \setminus \{\lambda_{i_j}\}$. So, the only element that is not a minimal generator of Λ and that may be a minimal generator of $\Lambda \setminus \{\lambda_{i_j}\}$ is $\lambda_{i_j} + \lambda_1$. $\square$

Lemma 4. *The ordinary semigroup $\Lambda = \{0, g+1, g+2, \ldots\}$ has minimal set of generators $\{g+1, g+2, \ldots, 2g+1\}$ and*

1. *$\Lambda \setminus \{g+1\}$ has $g+2$ minimal generators larger than its Frobenius number.*
2. *$\Lambda \setminus \{g+2\}$ has g minimal generators larger than its Frobenius number.*

3. $\Lambda \setminus \{g+r\}$, *with* $r > 2$, *has* $g-r+1$ *minimal generators larger than its Frobenius number.*

Proof. The first item is obvious.

As proved in Lemma 3 the only element that is not a minimal generator of Λ and that may be a minimal generator of $\Lambda \setminus \{\lambda_{i_j}\}$ is $\lambda_{i_j} + \lambda_1$. It is easy to prove that if $r = 2$ then $\lambda_{i_j} + \lambda_1$ is a minimal generator while if $r > 2$, it is not. □

Theorem 5. *The number* n_g *of numerical semigroups of genus* g *satisfies* $2F_g \leqslant n_g$ *for all* $g \geqslant 2$ *and* $2F_g \leqslant n_g \leqslant 1 + 3 \cdot 2^{g-3}$ *for all* $g \geqslant 3$.

Proof. Set $A_0 = B_0 = \{1\}$, $A_1 = B_1 = \{2\}$ and consider A_g and B_g defined as before for $g \geqslant 2$. Consider two trees A and B that respectively have A_g and B_g as the nodes at distance g from its root, with the element $a \in A_g$ having a children and the element $b \in B_g$ having b children. It is then easy to check that, by Lemma 3 and Lemma 4, the tree in Figure 3 contains A as a subtree and is contained in B. Thus, $|A_g| \leqslant n_g \leqslant |B_g|$. Now by Lemma 1 and Lemma 2 it follows the result. □

In Table 1 one can compare for g up to 30 the actual values of n_g with the bounds given in Theorem 5 and also with the bound given by the Catalan numbers proved in [3]. The values n_g are from [2].

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g	$2F_g$	n_g	$1+3\cdot 2^{g-3}$	C_g
0		1		1
1		1		1
2	2	2		2
3	4	4	4	5
4	6	7	7	14
5	10	12	13	42
6	16	23	25	132
7	26	39	49	429
8	42	67	97	1430
9	68	118	193	4862
10	110	204	385	16796
11	178	343	769	58786
12	288	592	1537	208012
13	466	1001	3073	742900
14	754	1693	6145	2674440
15	1220	2857	12289	9694845
16	1974	4806	24577	35357670
17	3194	8045	49153	129644790
18	5168	13467	98305	477638700
19	8362	22464	196609	1767263190
20	13530	37396	393217	6564120420
21	21892	62194	786433	24466267020
22	35422	103246	1572865	91482563640
23	57314	170963	3145729	343059613650
24	92736	282828	6291457	1289904147324
25	150050	467224	12582913	4861946401452
26	242786	770832	25165825	18367353072152
27	392836	1270267	50331649	69533550916004
28	635622	2091030	100663297	263747951750360
29	1028458	3437839	201326593	1002242216651368
30	1664080	5646773	402653185	3814986502092304

Table 1: Values of $2F_g$, n_g, $1+3\cdot 2^{g-3}$, and C_g for g up to 30.