Semigroup Forum (2008) 76: 379–384
DOI 10.1007/s00233-007-9014-8

RESEARCH PROBLEM

Fibonacci-like behavior of the number of numerical semigroups of a given genus

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Communicated by Jorge Almeida

Received 21 January 2007 and in final form 10 July 2007 / Published online: 19 October 2007

Abstract We conjecture a Fibonacci-like property on the number of numerical semigroups of a given genus. Moreover we conjecture that the associated quotient sequence approaches the golden ratio. The conjecture is motivated by the results on the number of semigroups of genus at most 50. The Wilf conjecture has also been checked for all numerical semigroups with genus in the same range.

Keywords Numerical semigroup · Fibonacci sequence · Genus · Wilf conjecture

1 Introduction

Let $\mathbb{N}_0$ denote the set of all non-negative integers. A *numerical semigroup* is a subset Λ of $\mathbb{N}_0$ containing 0, closed under addition and with finite complement in $\mathbb{N}_0$. For a numerical semigroup Λ define the *genus* of Λ as the number $g = \#(\mathbb{N}_0 \setminus \Lambda)$. As an example of numerical semigroup one can take

$$\{0, 4, 5, 8, 9, 10\} \cup \{i \in \mathbb{N}_0 : i \geq 12\}.$$

In this case the genus is 6.

We are interested on the number n_g of numerical semigoups of genus g. It is obvious that $n_0 = 1$ since $\mathbb{N}_0$ is the unique numerical semigroup of genus 0. On the other hand, if 1 is in a numerical semigroup, then any non-negative integer must belong also to the numerical semigroup, because any non-negative integer is a finite sum of 1's. Thus, the unique numerical semigroup with genus 1 is $\{0\} \cup \{i \in \mathbb{N}_0 : i \geq 2\}$ and $n_1 = 1$. We conjecture

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1. $n_g \geq n_{g-1} + n_{g-2}$, for $g \geq 2$
2. $\lim_{g\to\infty} \frac{n_{g-1}+n_{g-2}}{n_g} = 1$
3. $\lim_{g\to\infty} \frac{n_g}{n_{g-1}} = \phi$, where ϕ is the golden ratio.

Notice that point 2 would mean the sequence n_g to behave asymptotically as the Fibonacci sequence. This conjecture is motivated by the results on the number of semigroups of genus at most 50.

2 Computation of n_g

Every numerical semigroup can be generated by a finite set of elements and a minimal set of generators is unique (see for instance [3]). Let m be the smallest non-zero element of a numerical semigroup Λ, usually referred as its *multiplicity*. The set of minimal generators of Λ can be easily computed because it is a subset of the finite set

$$\{m\} \cup \{\lambda \in \Lambda \setminus \{0\} : \lambda - m \notin \Lambda\}.$$

This last set is intimately related to the Apéry set of Λ [1, 3–6].

The *conductor* of a numerical semigroup Λ is the unique integer c with $c - 1 \notin \Lambda$ and $c + \mathbb{N}_0 \subseteq \Lambda$. Given a numerical semigroup Λ of genus g and conductor c, $\Lambda \cup \{c-1\}$ is a numerical semigroup and its genus is $g-1$. So, any numerical semigroup of genus g can be obtained from a numerical semigroup of genus $g-1$ by removing one element larger than or equal to its conductor. It is easy to check that when removing such an element from a numerical semigroup, the set obtained is a numerical semigroup if and only if the removed element belongs to the set of minimal generators.

For instance, the unique numerical semigroup of genus 0 is $\mathbb{N}_0$. Its unique minimal generator is 1. Now, removing 1 from $\mathbb{N}_0$ we obtain

$$\Lambda_1 = \{0\} \cup \{i \in \mathbb{N}_0 : i \geq 2\}.$$

The numerical semigroup Λ_1 is the unique numerical semigroup of genus 1. In turn, Λ_1 has minimal generators 2 and 3. By removing 2 from Λ_1 we obtain the numerical semigroup

$$\Lambda_{2,1} = \{0\} \cup \{i \in \mathbb{N}_0 : i \geq 3\}$$

and by removing 3 from Λ_1 we obtain the numerical semigroup

$$\Lambda_{2,2} = \{0, 2\} \cup \{i \in \mathbb{N}_0 : i \geq 4\}.$$

The semigroups $\Lambda_{2,1}$ and $\Lambda_{2,2}$ are all the numerical semigroups of genus 2.

For the results in this paper we computed n_g by brute approach. That is, we generated all numerical semigroups of genus g from all numerical semigroups of genus $g-1$ as explained and then we counted them. This way one should be able to generate all semigroups of any given genus.

In the web page http://w3.impa.br/~nivaldo/algebra/semigroups/index.html by Nivaldo Medeiros one can find all numerical semigroups of genus up to 12. In Neil Sloane's On-line Encyclopedia of Integer Sequences [7] there are the values of n_g for $g \leq 14$. We could compute all numerical semigroups of genus up to 50.

The obstruction on the calculus of all numerical semigroups of a given *large* genus using the method explained above is the huge size of the results and the need to keep them for the next step. Indeed, the growth of n_g is apparently exponential with g and doing computations beyond a certain genus is really difficult using the computational means one can find at present. For instance, using a Pentium D 3.00 GHz with 1 GB of RAM it took 19 days to compute all semigroups of genus 50 and we expect that it would take about one month to compute all numerical semigroups of genus 51. The size of the compressed file containing all numerical semigroups of genus 50 is 3.6 GB.

In Table 1 there are the results obtained for all numerical semigroups with genus up to 50. For each genus we wrote the number of numerical semigroups of the given

Table 1 Computational results on the number of numerical semigroups up to genus 50

g	n_g	$n_{g-1}+n_{g-2}$	$\frac{n_{g-1}+n_{g-2}}{n_g}$	$\frac{n_g}{n_{g-1}}$
0	1			
1	1			1
2	2	2	1	2
3	4	3	0.75	2
4	7	6	0.857143	1.75
5	12	11	0.916667	1.71429
6	23	19	0.826087	1.91667
7	39	35	0.897436	1.69565
8	67	62	0.925373	1.71795
9	118	106	0.898305	1.76119
10	204	185	0.906863	1.72881
11	343	322	0.938776	1.68137
12	592	547	0.923986	1.72595
13	1001	935	0.934066	1.69088
14	1693	1593	0.940933	1.69131
15	2857	2694	0.942947	1.68754
16	4806	4550	0.946733	1.68218
17	8045	7663	0.952517	1.67395
18	13467	12851	0.954259	1.67396
19	22464	21512	0.957621	1.66808
20	37396	35931	0.960825	1.66471
21	62194	59860	0.962472	1.66312
22	103246	99590	0.964589	1.66006
23	170963	165440	0.967695	1.65588
24	282828	274209	0.969526	1.65432

Table 1 (*Continued*)

g	n_g	$n_{g-1}+n_{g-2}$	$\frac{n_{g-1}+n_{g-2}}{n_g}$	$\frac{n_g}{n_{g-1}}$
25	467224	453791	0.971249	1.65197
26	770832	750052	0.973042	1.64981
27	1270267	1238056	0.974642	1.64792
28	2091030	2041099	0.976121	1.64613
29	3437839	3361297	0.977735	1.64409
30	5646773	5528869	0.97912	1.64254
31	9266788	9084612	0.980341	1.64108
32	15195070	14913561	0.981474	1.63973
33	24896206	24461858	0.982554	1.63844
34	40761087	40091276	0.983567	1.63724
35	66687201	65657293	0.984556	1.63605
36	109032500	107448288	0.98547	1.63498
37	178158289	175719701	0.986312	1.63399
38	290939807	287190789	0.987114	1.63304
39	474851445	469098096	0.987884	1.63213
40	774614284	765791252	0.98861	1.63128
41	1262992840	1249465729	0.98929	1.63048
42	2058356522	2037607124	0.989919	1.62975
43	3353191846	3321349362	0.990504	1.62906
44	5460401576	5411548368	0.991053	1.62842
45	8888486816	8813593422	0.991574	1.62781
46	14463633648	14348888392	0.992067	1.62723
47	23527845502	23352120464	0.992531	1.62669
48	38260496374	37991479150	0.992969	1.62618
49	62200036752	61788341876	0.993381	1.6257
50	101090300128	100460533126	0.99377	1.62525

genus, the Fibonacci-like-estimated value given by the sum of the number of semigroups of the two previous genus, the value of the quotient $\frac{n_{g-1}+n_{g-2}}{n_g}$, and the value of the quotient $\frac{n_g}{n_{g-1}}$. In Fig. 1 and Fig. 2 we depicted the behavior of these quotients. From these graphics one can predict that $\frac{n_{g-1}+n_{g-2}}{n_g}$ approaches 1 as g approaches infinity whereas $\frac{n_g}{n_{g-1}}$ approaches the golden ratio as g approaches infinity. We leave this as a conjecture.

3 On the Wilf conjecture

The Wilf conjecture ([2, 8]) states that the number e of minimal generators of a numerical semigroup of genus g and conductor c satisfies

$$e \geq \frac{c}{c-g}.$$

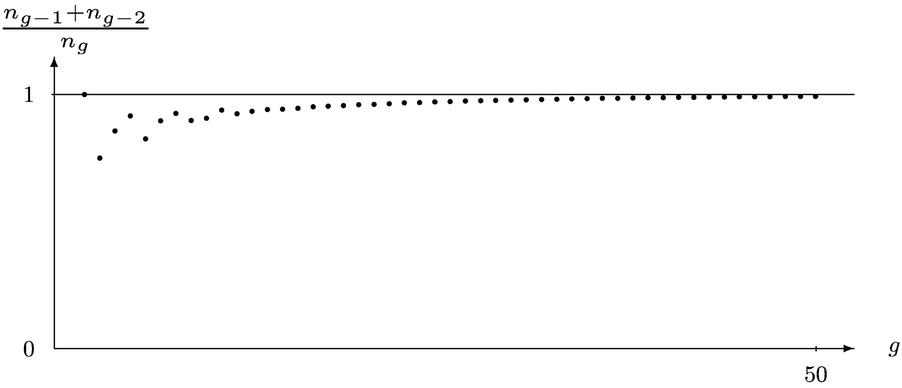

Fig. 1 Behavior of the quotient $\frac{n_{g-1}+n_{g-2}}{n_g}$. The values in this graphic correspond to the values in Table 1

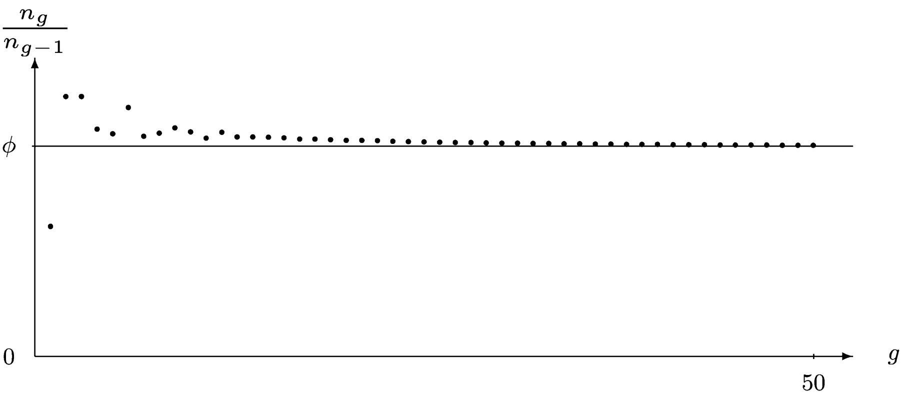

Fig. 2 Behavior of the quotient $\frac{n_g}{n_{g-1}}$. The values in this graphic correspond to the values in Table 1

It is easy to check it when the numerical semigroup is symmetric, that is, when $c = 2g$. In [2] the inequality is proved for many other cases. Here we proved by brute approach that any numerical semigroup of genus at most 50 also satisfies the conjecture.

Acknowledgements The author would like to thank Jordi Funollet and Josep M. Mondelo for their help on the computations and Pedro A. García-Sánchez, José Carlos Rosales, Aureliano M. Robles, and Ruud Pellikaan for many valuable and long discussions. She would also like to thank Anna de Mier, Raimon Elgueta, and Julio Fernández for their careful reading of the manuscript.

This work was partly supported by the Spanish Ministry of Science and Education through project CONSOLIDER INGENIO 2010 CSD2007-00004 "ARES".

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