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SHORT NOTE

Representation of Numerical Semigroups by Dyck Paths

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Abstract

We introduce square diagrams that represent numerical semigroups and we obtain an injection from the set of numerical semigroups into the set of Dyck paths.

Introduction

A *numerical semigroup* is a subset of $\mathbb{N}_0$ closed under addition and with finite complement in $\mathbb{N}_0$ [9], [5]. The *genus* of a numerical semigroup Λ is the number of elements in $\mathbb{N}_0 \setminus \Lambda$, which are called *gaps*. A *Dyck path* of order n is a lattice path from $(0,0)$ to (n,n) consisting of up-steps $\uparrow = (0,1)$ and right-steps $\rightarrow = (1,0)$ and never going below the diagonal $x = y$.

We introduce the notion of the *square diagram* of a numerical semigroup and analyze some properties of numerical semigroups such as their weight or symmetry by means of the square diagram.

We prove that any numerical semigroup is represented by a unique Dyck path of order given by its genus.

1. Square diagram

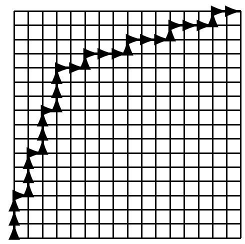
Given a numerical semigroup Λ define $\tau(\Lambda)$ as the path with origin $(0,0)$ and steps $e(i)$ given by

$$e(i) = \begin{cases} \rightarrow & \text{if } i \in \Lambda, \\ \uparrow & \text{if } i \notin \Lambda, \end{cases} \qquad \text{for } 1 \leqslant i \leqslant 2g.$$

We denote it as the *square diagram* of Λ. For instance, the set

$$\{0,4,8,12,16,17,19,20,21,23,24,25,27,28,29,31\} \cup \{i \in \mathbb{N}_0 \colon\ i \geqslant 31\}$$

is a numerical semigroup and its square diagram is the following one:



The *conductor* of a numerical semigroup is the unique element c in Λ such that $c-1 \notin \Lambda$ and $c+\mathbb{N}_0 \subseteq \Lambda$. The *enumeration* of Λ is the unique bijective increasing map λ: $\mathbb{N}_0 \longrightarrow \Lambda$. We use λ_i to denote $\lambda(i)$. The *ith partial genus* may be defined as $g(i) = \lambda_i - i = \#$ gaps smaller than λ_i.

Note that the following statements are satisfied:

(1) $g(0) = 0$,

(2) $g(i) \leqslant g(i+1)$,

(3) $g(i) = g$ for all $i \geqslant \lambda^{-1}(c)$,

(4) $g(i) = g$ for all $i \geqslant g$ (consequence of (3)).

The points with integer coordinates in the square diagram of Λ are all points in $\{(i, g(i))\colon\ 0 \leqslant i \leqslant g\} \cup \{(i-1, g(i))\colon\ 1 \leqslant i \leqslant g\}$ together with the points contained in the vertical lines from $(i-1, g(i-1))$ to $(i-1, g(i))$ whenever $g(i-1) < g(i)$. In particular, $\tau(\Lambda)$ goes from $(0,0)$ to (g,g). So it is included in the square grid from $(0,0)$ to (g,g). This is why we call this diagram the square diagram of Λ.

2. Square diagram of symmetric semigroups

The conductor c of any numerical semigroup satisfies $c \leqslant 2g$, where g is the genus of the semigroup. When $c = 2g$ the numerical semigroup is said to be *symmetric* [6], [5]. It is well known that all semigroups generated by two integers are symmetric. As a consequence of the definition of symmetric semigroups we have the following proposition.

Proposition 2.1. *A numerical semigroup Λ is symmetric if and only if its square diagram satisfies $e(2g-1) = \uparrow$.*

The next proposition is a well known result on symmetric semigroups.

Proposition 2.2. *A numerical semigroup Λ with conductor c is symmetric if and only if for any non-negative integer i, if i is a gap, then $c-1-i$ is a non-gap.*

The proof can be found in [6, Remark 4.2] and [5, Proposition 5.7]. It follows by counting the number of gaps and non-gaps smaller than the conductor and the fact that if i is a non-gap then $c-1-i$ must be a gap because otherwise

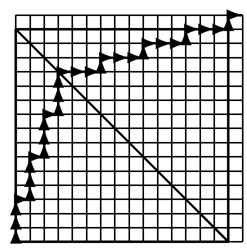
$c-1$ would also be a non-gap. As a consequence we have the following property on the square diagram of symmetric semigroups.

Corollary 2.3. *A numerical semigroup with genus g is symmetric if and only if the intersection of its square diagram with the square determined by the points $(0,0)$ and $(g-1,g-1)$ is symmetric with respect to the diagonal from $(0,g-1)$ to $(g-1,0)$.*

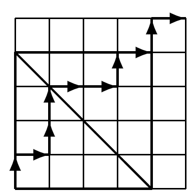
Example 2.4. The set

$$\{0,4,8,12,16,17,18,20,21,22,24,25,26,28,29,30,32\} \cup \{i \in \mathbb{N}_0\colon\ i \geqslant 32\}$$

is a numerical semigroup and it is symmetric. Its square diagram is therefore symmetric with respect to the diagonal from $(0,g-1)$ to $(g-1,0)$.



Remark 2.5. There exist paths from $(0,0)$ to $(g-1,g-1)$ which are symmetric with respect to the diagonal from $(0,g-1)$ to $(g-1,0)$ but which do not correspond to a numerical semigroup. For example,



This diagram does not correspond to a numerical semigroup because otherwise, λ_1 would be 2, but 4 would not belong to the semigroup.

3. Weight of a semigroup

The notion of the weight of a numerical semigroup has been widely used in the context of Weierstrass semigroups [8], [3].

Definition 3.1. Let Λ be a numerical semigroup with genus g and let $l_1,\dots,l_g$ be its gaps. The *weight* of Λ is the sum $\sum_{i=1}^{g}(l_i-i)$.

In a sense, the weight measures how complicated the semigroup is. For example, the simplest semigroup is that with gaps $1,2,\dots,g$ and it has weight 0.

Proposition 3.2. *The weight of a numerical semigroup is equal to the area over the path in the square diagram of the semigroup.*

Proof. Let λ be the enumeration of the semigroup. The area below the path is equal to the sum $\sum_{i=1}^{g} g(i)$, while the total area of the square diagram is g^2. So, it is enough to prove that

$$\sum_{i=1}^{g} g(i) + \sum_{i=1}^{g} (l_i - i) = g^2.$$

But $\sum_{i=1}^{g} g(i) + \sum_{i=1}^{g} (l_i - i) = \sum_{i=1}^{g} (\lambda_i - i) + \sum_{i=1}^{g} (l_i - i) = \sum_{i=1}^{2g} i - 2\sum_{i=1}^{g} i = \sum_{i=g+1}^{2g} i - \sum_{i=1}^{g} i = \sum_{i=1}^{g} (g+i) - \sum_{i=1}^{g} i = g^2$.

■

4. The square diagram of a numerical semigroup represents a Dyck path

Lemma 4.1. *Let Λ be a numerical semigroup with genus g, conductor c and enumeration λ. If $g(i) < i$, then*

(1) $\lambda_{i+1} = \lambda_i + 1$.

(2) $\lambda_i \geqslant c$.

Proof. (1) If $g(i) < i$, then there are more non-gaps than gaps in the interval $[1, \lambda_i]$. So by the Pigeonhole Principle there must be at least one pair $a, b \in \Lambda$ with $a + b = \lambda_i + 1$. Therefore, $\lambda_{i+1} = \lambda_i + 1$.

(2) Let us show by induction that $\lambda_i + k \in \Lambda$ for all $k \geqslant 0$. It is obvious for $k = 0$. If $\lambda_i + k' \in \Lambda$ for all $0 \leqslant k' \leqslant k$, then $g(i+k) = g(i) < i \leqslant i + k$ and, by (1), $\lambda_i + (k+1) \in \Lambda$. ■

Theorem 4.2. *The path $\tau(\Lambda)$ associated to a numerical semigroup Λ is a Dyck path.*

Proof. Let g and c be the genus and the conductor of Λ. Since $g(g) = g$, it is enough to show that $g(i) \geqslant i$ for all i with $0 \leqslant i < g$. Indeed, if $g(i) < i$, by Lemma 4.1, $\lambda_i \geqslant c$ and $g(i) = g$, so $i \geqslant g$, a contradiction. ■

Although it is not the goal of this note to find good bounds for the number of semigroups, we point out some enumerative consequences of Theorem 4.2. It is well known that the number of Dyck paths of order n is given by the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. This gives upper bounds on the number of numerical semigroups of a given genus. Unfortunately, they are quite far from the actual numbers. For instance, there are 37396 numerical semigroups of genus 20 (see [2]), while the bound given by the Catalan number is $C_{20} = 6564120420$.

On the other hand, the number of paths with m steps that start at $(0,0)$ and never go below the line $x = y$ is $\binom{m}{\lceil m/2 \rceil}$ (see [7]). Since symmetric semigroups are determined by the first half of the path, this gives the upper bound $\binom{g-1}{\lceil \frac{g-1}{2} \rceil}$ on the number of symmetric semigroups of genus g. Similarly, numerical semigroups with conductor c are determined by the first $c-2$ steps of the path. Thus the number of semigroups with conductor c is bounded above by $\binom{c-2}{\lceil \frac{c-2}{2} \rceil}$. Again, these bounds are very poor. Indeed, there are 227 symmetric semigroups of genus 20 (see [10]), while the corresponding bound is 92378, and there are 961 numerical semigroups with conductor 20 (see [10]), while the corresponding bound is 48620.

In fact, for the number of symmetric semigroups with genus g there is the upper bound $2^{\lfloor \frac{2g-1}{8} \rfloor}$ proved in [4, Proposition 5], whereas for the number of semigroups with conductor c, there is the upper bound $4 \cdot 2^{\lfloor \frac{c-2}{2} \rfloor}$ proved in [1]. In both cases they are sharper bounds than the ones just deduced from Theorem 4.2.

A very attractive open problem is to characterize combinatorially the subset of Dyck paths that correspond to numerical semigroups. In addition to its intrinsic interest, its solution might help improve these bounds.

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References

[1] Backelin, J., *On the number of semigroups of natural numbers*, Math. Scand. **66**(2) (1990), 197–215.

[2] Bras-Amorós, M., *Fibonacci-like behavior of the number of semigroups of a given genus*, Semigroup Forum (submitted).

[3] Carvalho, C. and F. Torres, *On numerical semigroups related to covering of curves*, Semigroup Forum **67**(3) (2003), 344–354.

[4] Fröberg, R., C. Gottlieb and R. Häggkvist, *On numerical semigroups*, Semigroup Forum **35**(1) (1987), 63–83.

[5] Høholdt, T., J. H. van Lint and R. Pellikaan, "Algebraic Geometry Codes", Handbook of Coding Theory (1), pp. 871–961, North-Holland, Amsterdam, 1998.

[6] Kirfel, C. and R. Pellikaan, *The minimum distance of codes in an array coming from telescopic semigroups*, IEEE Trans. Inform. Theory **41**(6, part 1) (1995), 1720–1732 (special issue on algebraic geometry codes).

[7] Mohanty, S. G., "Lattice Path Counting and Applications", Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1979.

[8] Oliveira, G., *Weierstrass semigroups and the canonical ideal of nontrigonal curves*, Manuscripta Math. **71**(4) (1991), 431–450.

[9] Rosales, J. C. and P. A. García-Sánchez, "Finitely Generated Commutative Monoids", Nova Science Publishers Inc., Commack, NY, 1999.

[10] Rosales, J. C., P. A. García-Sánchez, J. I. García-García and J. A. Jiménez Madrid, *Fundamental gaps in numerical semigroups*, J. Pure Appl. Algebra **189**(1–3) (2004), 301–313.

[11] Stanley, R. P., "Enumerative Combinatorics", Vol. 2, volume 62 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and Appendix 1 by Sergey Fomin.

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