Acta Mathematica Sinica, English Series
Jan., 2011, Vol. 27, No. 1, pp. 73–82
Published online: December 15, 2010
DOI: 10.1007/s10114-011-9075-1
Http://www.ActaMath.com

Acta Mathematica Sinica, English Series

Partitioning a Graph into Defensive *k*-Alliances

Ismael G. YERO
Department of Computer Engineering and Mathematics,
Rovira i Virgili University, Av. Països Catalans 26, 43007 *Tarragona, Spain*
E-mail: *ismael.gonzalez@urv.cat*

Sergio BERMUDO
Department of Economy, Quantitative Methods and Economic History,
Pablo de Olavide University, Carretera de Utrera Km. 1, 41013-*Sevilla, Spain*
E-mail: *sbernav@upo.es*

Juan A. RODRíGUEZ-VELÁZQUEZ
Department of Computer Engineering and Mathematics,
Rovira i Virgili University, Av. Països Catalans 26, 43007 *Tarragona, Spain*
E-mail: *juanalberto.rodriguez@urv.cat*

José M. SIGARRETA
Faculty of Mathematics, Autonomous University of Guerrero,
Carlos E. Adame 5, *Col. La Garita, Acapulco, Guerrero, México*
E-mail: *josemariasigarretaalmira@hotmail.com*

Abstract A defensive k-alliance in a graph is a set S of vertices with the property that every vertex in S has at least k more neighbors in S than it has outside of S. A defensive k-alliance S is called global if it forms a dominating set. In this paper we study the problem of partitioning the vertex set of a graph into (global) defensive k-alliances. The (global) defensive k-alliance partition number of a graph $\Gamma=(V,E)$, $(\psi_k^{gd}(\Gamma))$ $\psi_k^{d}(\Gamma)$, is defined to be the maximum number of sets in a partition of V such that each set is a (global) defensive k-alliance. We obtain tight bounds on $\psi_k^{d}(\Gamma)$ and $\psi_k^{gd}(\Gamma)$ in terms of several parameters of the graph including the order, size, maximum and minimum degree, the algebraic connectivity and the isoperimetric number. Moreover, we study the close relationships that exist among partitions of $\Gamma_1\times\Gamma_2$ into (global) defensive (k_1+k_2)-alliances and partitions of Γ_i into (global) defensive k_i-alliances, $i\in\{1,2\}$.

Keywords Defensive alliances, dominating sets, domination, isoperimetric number

MR(2000) Subject Classification 05C69, 05C70

Received February 11, 2009, accepted October 27, 2009
Supported by the Spanish Ministry of Science and Innovation through Projects TSI2007-65406-C03-01 "E-AEGIS", CONSOLIDER INGENIO 2010 CSD2007-0004 "ARES" and MTM2009-09501, by the Rovira i Virgili University through Project 2006AIRE-09 and by the Junta de Andalucía, ref. FQM-260 and ref. P06-FQM-02225

1 Introduction

Since (defensive, offensive and dual) alliances in graph were first introduced by Kristiansen, et al. [1], several authors have studied their mathematical properties [2–17]. We are interested in a generalization of defensive alliances, called k-alliances, introduced by Shafique and Dutton in [12, 13]. We focus our attention on the problem of partitioning the vertex set of a graph into defensive k-alliances. This problem has been previously studied by Shafique and Dutton [14, 18], and the particular case $k = -1$ has been studied by Eroh and Gera [19, 20] and by Haynes and Lachniet [21].

We begin by stating the terminology used. Throughout this article, $\Gamma = (V, E)$ denotes a simple graph of order $|V| = n$ and size $|E| = m$. We denote two adjacent vertices u and v by $u \sim v$, the degree of a vertex $v \in V$ by $\delta(v)$, the minimum degree by δ and the maximum degree by Δ. For a nonempty set $X \subseteq V$, and a vertex $v \in V$, $N_X(v)$ denotes the set of neighbors v has in X: $N_X(v) := \{u \in X : u \sim v\}$, and the degree of v in X will be denoted by $\delta_X(v) = |N_X(v)|$. The subgraph induced by $S \subset V$ will be denoted by $\langle S \rangle$, and the complement of the set S in V will be denoted by $\bar{S}$.

A nonempty set $S \subseteq V$ is a *defensive k-alliance* in $\Gamma = (V, E)$, $k \in \{-\Delta, \ldots, \Delta\}$, if for every $v \in S$,

$$\delta_S(v) \geq \delta_{\bar{S}}(v) + k. \tag{1.1}$$

Notice that (1.1) is equivalent to

$$\delta(v) \geq 2\delta_{\bar{S}}(v) + k.$$

For example, if $k > 1$, the star graph $K_{1,t}$ has no defensive k-alliances, and every set composed by two adjacent vertices in a cubic graph is a defensive (-1)-alliance. For graphs having defensive k-alliances, the *defensive k-alliance number* of Γ, denoted by $a_k^d(\Gamma)$, is defined as the minimum cardinality of a defensive k-alliance in Γ. Notice that

$$a_{k+1}^d(\Gamma) \geq a_k^d(\Gamma).$$

For the study of the mathematical properties of $a_k^d(\Gamma)$, we cite [10].

A set $S \subset V$ is a *dominating set* in $\Gamma = (V, E)$ if for every vertex $u \in \bar{S}$, $\delta_S(u) > 0$ (every vertex in $\bar{S}$ is adjacent to at least one vertex in S). The *domination number* of Γ, denoted by $\gamma(\Gamma)$, is the minimum cardinality of a dominating set in Γ.

A defensive k-alliance S is called *global* if it forms a dominating set. For graphs having global defensive k-alliances, the *global defensive k-alliance number* of Γ, denoted by $\gamma_k^d(\Gamma)$, is the minimum cardinality of a global defensive k-alliance in Γ. Clearly,

$$\gamma_{k+1}^d(\Gamma) \geq \gamma_k^d(\Gamma) \geq \gamma(\Gamma) \quad \text{and} \quad \gamma_k^d(\Gamma) \geq a_k^d(\Gamma).$$

For the study of the mathematical properties of $\gamma_k^d(\Gamma)$, we cite [11].

The (*global*) *defensive k-alliance partition number* of Γ, $(\psi_k^{gd}(\Gamma))$ $\psi_k^d(\Gamma)$, $k \in \{-\Delta, \ldots, \delta\}$, is defined to be the maximum number of sets in a partition of $V(\Gamma)$ such that each set is a (global) defensive k-alliance. Extreme cases are $\psi_{-\Delta}^d(\Gamma) = n$, where each set composed of one vertex is a defensive $(-\Delta)$-alliance, and $\psi_\delta^d(\Gamma) = 1$ for the case of a connected δ-regular graph where $V(\Gamma)$ is the only defensive δ-alliance. A graph Γ is *partitionable* into (global) defensive

k-alliances if ($\psi_k^{gd}(\Gamma) \geq 2$) $\psi_k^d(\Gamma) \geq 2$. Hereafter we will say that $\Pi_r(\Gamma) = \{V_1, V_2, \ldots, V_r\}$ is a partition of Γ into r (global) defensive k-alliances.

Notice that if every vertex of Γ has even degree and k is odd, $k = 2l-1$, then every (global) defensive $(2l-1)$-alliance in Γ is a (global) defensive $(2l)$-alliance and vice versa. Hence, in such a case, $a_{2l-1}^d(\Gamma) = a_{2l}^d(\Gamma)$, $\Gamma_{2l-1}^d(\Gamma) = \gamma_{2l}^d(\Gamma)$, $\psi_{2l-1}^d(\Gamma) = \psi_{2l}^d(\Gamma)$ and $\psi_{2l-1}^{gd}(\Gamma) = \psi_{2l}^{gd}(\Gamma)$.

Analogously, if every vertex of Γ has odd degree and k is even, $k = 2l$, then every defensive $(2l)$-alliance in Γ is a defensive $(2l+1)$-alliance and vice versa. Hence, in such a case, $a_{2l}^d(\Gamma) = a_{2l+1}^d(\Gamma)$, $\gamma_{2l}^d(\Gamma) = \gamma_{2l+1}^d(\Gamma)$, $\psi_{2l}^d(\Gamma) = \psi_{2l+1}^d(\Gamma)$ and $\psi_{2l}^{gd}(\Gamma) = \psi_{2l+1}^{gd}(\Gamma)$.

2 Partitioning a Graph into Defensive *k*-Alliances

Example 1 Let k and r be integers such that $r > 1$ and $r + k > 0$, and let $\mathcal{H}$ be a family of graphs whose vertex set is $V = \bigcup_{i=1}^r V_i$ where, for every V_i, $\langle V_i \rangle \cong K_{r+k}$ and $\delta_{V_j}(v) = 1$, for every $v \in V_i$ and $j \neq i$. Notice that $\{V_1, V_2, \ldots, V_r\}$ is a partition of the graphs belonging to $\mathcal{H}$ into r global defensive k-alliances. A particular family of graphs included in $\mathcal{H}$ is $K_{r+k} \times K_r$.

Hereafter, $\mathcal{H}$ will denote the family of graphs defined in the above example.

From the following relation between the defensive k-alliance numbers, $a_k^d(\Gamma)$ and $\psi_k^d(\Gamma)$, we obtain that lower bounds on $a_k^d(\Gamma)$ lead to upper bounds on $\psi_k^d(\Gamma)$:

$$a_k^d(\Gamma)\psi_k^d(\Gamma) \leq n. \tag{2.1}$$

For instance, it was shown in [10] that

$$a_k^d(\Gamma) \geq \left\lceil \frac{\delta + k + 2}{2} \right\rceil. \tag{2.2}$$

An example of equality in the above bound is provided by the graphs belonging to the family $\mathcal{H}$, for which we obtain $a_k^d(\Gamma) = r + k$.

By (2.1) and (2.2) we obtain the following bound:

$$\psi_k^d(\Gamma) \leq \begin{cases} \left\lfloor \dfrac{2n}{\delta + k + 2} \right\rfloor, & \delta + k \text{ even}, \\ \left\lfloor \dfrac{2n}{\delta + k + 3} \right\rfloor, & \delta + k \text{ odd}. \end{cases}$$

This bound gives the exact value of $\psi_k^d(\Gamma)$, for instance, for every $\Gamma \in \mathcal{H}$, where $\psi_k^d(\Gamma) = r$, and in the following cases: $\psi_{-1}^d(K_4 \times C_4) = 5$, $\psi_0^d(K_3 \times C_4) = \psi_{-1}^d(K_2 \times C_4) = 4$ and $\psi_1^d(K_2 \times C_4) = 2$.

Analogously, for global alliances we have

$$\gamma_k^d(\Gamma)\psi_k^{gd}(\Gamma) \leq n. \tag{2.3}$$

One example of bounds on $\gamma_k^d(\Gamma)$, obtained in [11], is the following

$$\gamma_k^d(\Gamma) \geq \left\lceil \frac{n}{\left\lfloor \frac{\Delta - k}{2} \right\rfloor + 1} \right\rceil. \tag{2.4}$$

For the graphs in $\mathcal{H}$, the above bound gives the exact value $\gamma_k^d(\Gamma) = r + k$. Thus, the bound obtained by combining (2.3) and (2.4),

$$\psi_k^{gd}(\Gamma) \leq \left\lfloor \frac{\Delta - k}{2} \right\rfloor + 1,$$

leads to the exact value of $\psi_k^{gd}(\Gamma) = r$ for every $\Gamma \in \mathcal{H}$. Even so, this bound can be improved.

Theorem 2.1 *For every graph* Γ *partitionable into global defensive* k*-alliances,*

(i) $\psi_k^{gd}(\Gamma) \le \lfloor \frac{\sqrt{k^2+4n}-k}{2} \rfloor$,

(ii) $\psi_k^{gd}(\Gamma) \le \lfloor \frac{\delta-k+2}{2} \rfloor$.

Proof Since, every $V_i \in \Pi_r(\Gamma)$ is a dominating set, we have that for every $v \in V_i$, $\delta_{\overline{V_i}}(v) \ge r-1$. Thus, the bounds are obtained as follows:

(i) $|V_i| - 1 \ge \delta_{V_i}(v) \ge \delta_{\overline{V_i}}(v) + k \ge r - 1 + k$, so $n = \sum_{i=1}^r |V_i| \ge r(r+k)$. By solving the inequality $r^2 + kr - n \le 0$, we obtain the result.

(ii) Taking $v \in V_i$ as a vertex of minimum degree, we obtain the result from $\delta = \delta(v) \ge 2\delta_{\overline{V_i}}(v) + k \ge 2(r-1) + k$. □

The above bounds are attained, for instance, in the following cases: $\psi_{-1}^{gd}(K_4 \times C_4) = 4$, $\psi_0^{gd}(K_3 \times C_4) = 3$, $\psi_1^{gd}(K_2 \times C_4) = 2$ and $\psi_1^{gd}(P) = 2$, where P denotes the Petersen graph.

Remark 2.2 For every $k \in \{1-\delta, \ldots, \delta\}$, if $\psi_k^{gd}(\Gamma) \ge 2$, then

$$\gamma_k^d(\Gamma) + \psi_k^{gd}(\Gamma) \le \frac{n+4}{2}.$$

Proof By (2.3) we have $\gamma_k^d(\Gamma) + \psi_k^{gd}(\Gamma) \le \frac{n+\left(\psi_k^{gd}(\Gamma)\right)^2}{\psi_k^{gd}(\Gamma)}$. On the other hand, if $k \in \{1-\delta, \ldots, \delta\}$, then $\gamma_k^d(\Gamma) \ge 2$. Moreover, if $\psi_k^{gd}(\Gamma) \ge 2$, then $\gamma_k^d(\Gamma) \le \frac{n}{2}$. So, $2 \le \psi_k^{gd}(\Gamma) \le \frac{n}{\gamma_k^d(\Gamma)} \le \frac{n}{2}$. As a consequence, the result is obtained as follows:

$$\max_{2 \le x \le \frac{n}{\gamma_k^d(\Gamma)}} \left\{ \frac{n+x^2}{x} \right\} = \max\left\{ \frac{n+4}{2}, \frac{n+(\gamma_k^d(\Gamma))^2}{\gamma_k^d(\Gamma)} \right\} = \frac{n+4}{2}. \qquad \square$$

Example of equality in the above bound is $\gamma_{-1}^d(C_4 \times K_2) + \psi_{-1}^{gd}(C_4 \times K_2) = 6$.

Theorem 2.3 *Let* $C_{(r,k)}^{gd}(\Gamma)$ *be the minimum number of edges having its endpoints in different sets of a partition of* Γ *into* $r \ge 2$ *global defensive* k*-alliances. Then*

(i) $C_{(r,k)}^{gd}(\Gamma) \ge \frac{1}{2} r(r-1)\gamma_k^d(\Gamma)$,

(ii) $C_{(r,k)}^{gd}(\Gamma) \ge \frac{1}{2} r(r-1)(r+k)$,

(iii) $C_{(r,k)}^{gd}(\Gamma) \le \frac{2m-nk}{4}$,

(iv) $C_{(r,k)}^{gd}(\Gamma) = \frac{1}{2} r(r-1)\gamma_k^d(\Gamma) = \frac{1}{2} r(r-1)(r+k) = \frac{2m-nk}{4}$ *if and only if* $\Gamma \in \mathcal{H}$.

Proof Let $x = \min_{V_i \in \Pi_r(\Gamma)} |V_i|$. From the fact that every set of $\Pi_r(\Gamma)$ is a dominating set, we obtain that the number of edges adjacent to $v \in V_i$ with one endpoint in $\bigcup_{j=i+1}^r V_j$ is bounded by $\sum_{j=i+1}^r \delta_{V_j}(v) \ge r - i$. Therefore,

$$C_{(r,k)}^{gd}(\Gamma) \ge \sum_{i=1}^{r-1} (r-i)|V_i| \ge x \sum_{i=1}^{r-1} (r-i) = \frac{x}{2} r(r-1). \qquad (2.5)$$

Since every $V_i \in \Pi_r(\Gamma)$ is a global defensive k-alliance, we have $x \ge r+k$ and $x \ge \gamma_k^d(\Gamma)$; as a consequence, (i) and (ii) follow.

In order to obtain the upper bound in (iii) we note that the number of edges in Γ with one endpoint in V_i and the other endpoint in V_j is $C(V_i, V_j) = \sum_{v \in V_i} \delta_{V_j}(v) = \sum_{v \in V_j} \delta_{V_i}(v)$. Hence,

$$2m = \sum_{i=1}^r \sum_{v \in V_i} \delta(v) \ge 2 \sum_{i=1}^r \sum_{v \in V_i} \delta_{\overline{V_i}}(v) + k \sum_{i=1}^r |V_i|$$

$$= 2\sum_{i=1}^{r}\sum_{v\in V_i}\sum_{j=1,j\neq i}^{r}\delta_{V_j}(v) + kn$$
$$= 2\sum_{i=1}^{r}\sum_{j=1,j\neq i}^{r}\sum_{v\in V_i}\delta_{V_j}(v) + kn$$
$$= 2\sum_{i=1}^{r}\sum_{j=1,j\neq i}^{r}C(V_i,V_j) + nk$$
$$= 4C^{gd}_{(r,k)}(\Gamma) + nk.$$

Therefore, (iii) follows.

If for some $V_i \in \Pi_r(\Gamma)$, there exists $v \in V_i$ such that $\delta_{V_i}(v) > \delta_{\overline{V_i}}(v) + k$, then, by analogy to the proof of (iii) we obtain $C^{gd}_{(r,k)}(\Gamma) < \frac{2m-nk}{4}$. Therefore, if $C^{gd}_{(r,k)}(\Gamma) = \frac{2m-nk}{4}$, then for every $V_i \in \Pi_r(\Gamma)$, and for every $v \in V_i$, we have

$$\delta_{V_i}(v) = \delta_{\overline{V_i}}(v) + k. \tag{2.6}$$

Moreover, if for some $V_i \in \Pi_r(\Gamma)$ there exists $v \in V_i$ such that $\sum_{j\neq i}\delta_{V_i}(v) > r-1$, then, by analogy to the proof of (i) and (ii) we obtain $C^{gd}_{(r,k)}(\Gamma) > \frac{1}{2}r(r-1)\gamma^d_k(\Gamma)$ and $C^{gd}_{(r,k)}(\Gamma) > \frac{1}{2}r(r-1)(r+k)$. Therefore, if $C^{gd}_{(r,k)}(\Gamma) = \frac{1}{2}r(r-1)\gamma^d_k(\Gamma) = \frac{1}{2}r(r-1)(r+k)$, then for every $V_i \in \Pi_r(\Gamma)$, and for every $v \in V_i$, we have

$$\delta_{\overline{V_i}}(v) = \sum_{j\neq i}\delta_{V_i}(v) = r-1. \tag{2.7}$$

So, by (2.6) and (2.7) we obtain that for every $V_i \in \Pi_r(\Gamma)$, $\langle V_i\rangle$ is regular of degree $r+k-1$. Thus, Γ is a regular graph of degree $2(r-1)+k$ and, by $\frac{1}{2}r(r-1)\gamma^d_k(\Gamma) = \frac{1}{2}r(r-1)(r+k) = \frac{2m-nk}{4}$ we have $n(\Gamma) = r(r+k)$ and $\gamma^d_k(\Gamma) = r+k$. Hence, $|V_i| = r+k$, so $\langle V_i\rangle \cong K_{r+k}$. Moreover, as every $V_j \in \Pi_r(\Gamma)$ is a dominating set, by (2.7) we have $\delta_{V_j}(v) = 1$, for every $v \in V_i$, $i \neq j$. Therefore, $\Gamma \in \mathcal{H}$. The opposite implication is immediate. □

By (2.5) and Theorem 2.3 (iii), we obtain the following result:

Corollary 2.4 *For every graph Γ partitionable into r global defensive k-alliances of equal cardinality, $r \leq \frac{2(m+n)-kn}{2n}$.*

A family of graphs that achieve equality for Corollary 2.4 is the family $\mathcal{H}$ defined in Example 1.

By Theorem 2.3 and (2.2) we obtain the following two necessary conditions for the existence of a partition of a graph into r global defensive k-alliances.

Corollary 2.5 *If for a graph Γ, $k > \frac{2m-r(r-1)(\delta+2)}{n+r(r-1)}$ or $k > \frac{2(m-r^2(r-1))}{n+2r(r-1)}$, then Γ cannot be partitioned into r global defensive k-alliances.*

By the above corollary we conclude, for instance, that the 3-cube graph cannot be partitioned into $r > 2$ global defensive k-alliances.

Remark 2.6 The size of the subgraph induced by a set belonging to a partition of Γ into r global defensive k-alliances is bounded below by $\frac{1}{2}\gamma^d_k(\Gamma)(r+k-1)$.

Proof The result follows from the fact that for every $V_i \in \Pi_r(\Gamma)$,

$$\sum_{v\in V_i} \delta_{V_i}(v) \geq ((r-1)+k)|V_i| \geq (r-1+k)\gamma_k^d(\Gamma). \qquad \square$$

The above bound is tight as we can check by taking $\Gamma \in \mathcal{H}$.

2.1 Isoperimetric Number, Bisection and k-Alliances

The *isoperimetric number* of a graph $\Gamma = (V, E)$, defined as

$$\mathbf{i}(\Gamma) := \min_{S\subset V(\Gamma):|S|\leq \frac{n}{2}} \left\{ \frac{\sum_{v\in S} \delta_{\overline{S}}(v)}{|S|} \right\},$$

has been extensively studied. For instance, we cite the papers by Mohar [22], Kahale [23] and Kwak et al. [24]. This graph invariant is very hard to compute, and even obtaining bounds on $\mathbf{i}(\Gamma)$ is not straightforward. Here we consider the case of graphs which are partitionable into defensive k-alliances and, for these graphs, we obtain a tight bound on $\mathbf{i}(\Gamma)$.

As a consequence of Theorem 2.3 (iii), we obtain the following result.

Corollary 2.7 *If there exists a partition Π_r of Γ into $r \geq 2$ global defensive k-alliances such that, for every $V_i \in \Pi_r$, $|V_i| \leq \frac{n}{2}$, then*

$$\mathbf{i}(\Gamma) \leq \frac{2m - nk}{2n}.$$

Proof For every $V_i \in \Pi_r$ we have $|V_i|\mathbf{i}(\Gamma) \leq \sum_{v\in V_i} \delta_{\overline{V_i}}(v) = \sum_{v\in V_i}\sum_{j=1,j\neq i}^{r} \delta_{V_j}(v)$. Hence,

$$n\mathbf{i}(\Gamma) = \mathbf{i}(\Gamma)\sum_{i=1}^{r}|V_i| \leq \sum_{i=1}^{r}\sum_{v\in V_i}\sum_{j=1,j\neq i}^{r} \delta_{V_j}(v) = 2C_{(r,k)}^{gd}(\Gamma) \leq \frac{2m-nk}{2}. \qquad \square$$

Example of equality in the above bound is the graph $\Gamma = C_3 \times C_3$ for $k = 0$. That is, $C_3 \times C_3$ can be partitioned into $r = 3$ global defensive 0-alliances of cardinality 3; moreover, $\mathbf{i}(C_3 \times C_3) = 2$. Other example is the 3-cube graph $\Gamma = C_4 \times K_2$, for $k = 1$. In this case, each copy of the cycle C_4 is a global defensive 1-alliance and $\mathbf{i}(C_4 \times K_2) = 1$.

Notice that if $\mathbf{i}(\Gamma) > \frac{2m-nk}{2n}$, then Γ cannot be partitioned into $r \geq 2$ global defensive k-alliances with the condition that the cardinality of every set in the partition is at most $\frac{n}{2}$. One example of this is the graph $\Gamma = C_3 \times C_3$ for $k \geq 1$.

Theorem 2.8 *For any graph Γ,*

(i) *if Γ is partitionable into global defensive k-alliances, then*

$$\psi_k^{gd}(\Gamma) \leq \Delta + 1 - \mathbf{i}(\Gamma) - k,$$

(ii) *if Γ is partitionable into defensive k-alliances, then*

$$a_k^d(\Gamma) \geq \mathbf{i}(\Gamma) + k + 1.$$

Proof (i) Let $\Pi_r(\Gamma)$ be a partition of Γ into $r \geq 2$ global defensive k-alliances. Then, there exists $V_i \in \Pi_r(\Gamma)$ such that $|V_i| \leq \frac{n}{2}$. Hence,

$$|V_i|\mathbf{i}(\Gamma) \leq \sum_{v\in V_i} \delta_{\overline{V_i}}(v) \leq \sum_{v\in V_i}(\delta_{V_i}(v) - k) \leq \sum_{v\in V_i}(\delta(v) - r + 1 - k) \leq |V_i|(\Delta - r + 1 - k).$$

Thus, $r \leq \Delta + 1 - \mathbf{i}(\Gamma) - k$.

(ii) If $\psi_k^d(\Gamma) \geq 2$, then there exists a defensive k-alliance S such that $|S| \leq \frac{n}{2}$. Therefore,

$$|S|\mathrm{i}(\Gamma) \leq \sum_{v\in S} \delta_{\overline{S}}(v) \leq \sum_{v\in S}(\delta_S(v) - k) \leq |S|(|S|-1) - k|S|.$$

Thus, the result follows. □

The following relation between the algebraic connectivity and the isoperimetric number of a graph was shown by Mohar in [22]: $\mathbf{i}(\Gamma) \geq \frac{\mu}{2}$.

Corollary 2.9 *For any graph* Γ,

(i) *if* Γ *is partitionable into global defensive* k*-alliances, then*

$$\psi_k^{gd}(\Gamma) \leq \left\lfloor \Delta + 1 - \frac{\mu}{2} - k \right\rfloor,$$

(ii) *if* Γ *is partitionable into defensive* k*-alliances, then*

$$a_k^d(\Gamma) \geq \left\lceil \frac{\mu + 2(k+1)}{2} \right\rceil.$$

Example of equality in the above bounds is the graph $\Gamma = C_3 \times C_3$ for $k = 0$, in this case $\mu = 3$.

From the above corollary, we emphasize that if $\mu > 2(\Delta - 1 - k)$, then Γ cannot be partitioned into global defensive k-alliances. For instance, we conclude that $\Gamma = C_3 \times C_3$ cannot be partitioned into global defensive k-alliances for $k > 1$. Moreover, by Corollary 2.9 (ii) we conclude that if $a_k^d(\Gamma) < \lceil \frac{\mu+2(k+1)}{2} \rceil$, then Γ cannot be partitioned into defensive k-alliances.

A *bisection* of Γ is a 2-partition $\{X, Y\}$ of the vertex set $V(\Gamma)$ in which $|X| = |Y|$ or $|X| = |Y| + 1$. The bisection problem is to find a bisection for which $\sum_{v\in X} \delta_Y(v)$ is as small as possible. The *bipartition width*, $bw(\Gamma)$, is defined as

$$bw(\Gamma) := \min_{X\subset V(\Gamma), |X| = \lfloor \frac{n}{2} \rfloor} \left\{ \sum_{v\in X} \delta_{\overline{X}}(v) \right\}.$$

It was shown by Merris [25] and Mohar [22] that

$$bw(\Gamma) \geq \begin{cases} \left\lceil \dfrac{n\mu}{4} \right\rceil, & \text{if } n \text{ is even,} \\ \left\lceil \dfrac{(n^2-1)\mu}{4n} \right\rceil, & \text{if } n \text{ is odd.} \end{cases}$$

We are interested in the bisection of a graph into global defensive k-alliances, i.e., the bisection $\{X, Y\}$ of V such that X and Y are global defensive k-alliances. An example of bisection into global defensive $(t-1)$-alliances is obtained for the family of hypercube graphs $Q_{t+1} = Q_t \times K_2$, by taking $\{X, Y\}$ such that $\langle X \rangle \cong Q_t \cong \langle Y \rangle$.

By Theorem 2.3 (iii) and the above bound we obtain the following result.

Corollary 2.10 *If* $\lfloor \frac{2m-nk}{4} \rfloor < \lceil \frac{n\mu}{4} \rceil$, *for* n *even; or* $\lfloor \frac{2m-nk}{4} \rfloor < \lceil \frac{(n^2-1)\mu}{4n} \rceil$, *for* n *odd, then* Γ *cannot be bisectioned into global defensive* k*-alliances.*

For example, according to Corollary 2.10 we can conclude that, for $k > 0$, the graph $C_3 \times C_3$ cannot be bisectioned into global defensive k-alliances.

3 Partitioning $\Gamma_1 \times \Gamma_2$ into (Global) Defensive k-Alliances

In Subsection 3.1 we will discuss the close relationships that exist among $\psi^d_{k_1+k_2}(\Gamma_1 \times \Gamma_2)$ and $\psi^d_{k_i}(\Gamma_i)$, $i \in \{1, 2\}$. Obviously, we begin with the study of the relationship between $a^d_{k_1+k_2}(\Gamma_1 \times \Gamma_2)$ and $a^d_{k_i}(\Gamma_i)$, $i \in \{1, 2\}$. The case of global alliances will be studied in Subsection 3.2.

3.1 Partitioning $\Gamma_1 \times \Gamma_2$ into Defensive k-Alliances

Theorem 3.1 *For any graphs Γ_1 and Γ_2,*

(i) *if Γ_i contains a defensive k_i-alliance, $i \in \{1, 2\}$, then $\Gamma_1 \times \Gamma_2$ contains a defensive $(k_1 + k_2)$-alliance and*

$$a^d_{k_1+k_2}(\Gamma_1 \times \Gamma_2) \leq a^d_{k_1}(\Gamma_1)a^d_{k_2}(\Gamma_2),$$

(ii) *if there exists a partition of Γ_i into defensive k_i-alliances, $i \in \{1, 2\}$, then there exists a partition of $\Gamma_1 \times \Gamma_2$ into defensive $(k_1 + k_2)$-alliances and*

$$\psi^d_{k_1+k_2}(\Gamma_1 \times \Gamma_2) \geq \psi^d_{k_1}(\Gamma_1)\psi^d_{k_2}(\Gamma_2).$$

Proof Let S_i be a defensive k_i-alliance in Γ_i, $i \in \{1, 2\}$, and let $X = S_1 \times S_2$. Then for every $x = (u, v) \in X$,

$$\begin{aligned} \delta_X(x) &= \delta_{S_1}(u) + \delta_{S_2}(v) \\ &\geq \big(\delta_{\bar{S}_1}(u) + k_1\big) + \big(\delta_{\bar{S}_2}(v) + k_2\big) \\ &= \delta_{\bar{X}}(x) + k_1 + k_2. \end{aligned}$$

Thus, X is a defensive (k_1+k_2)-alliance in $\Gamma_1 \times \Gamma_2$ and, as a consequence, (i) follows. Moreover, we conclude that every partition

$$\Pi_{r_i}(\Gamma_i) = \{S^{(i)}_1, S^{(i)}_2, \ldots, S^{(i)}_{r_i}\}$$

of Γ_i into r_i defensive k_i-alliances induces a partition of $\Gamma_1 \times \Gamma_2$ into r_1r_2 defensive $(k_1 + k_2)$-alliances:

$$\Pi_{r_1r_2}(\Gamma_1 \times \Gamma_2) = \left\{ \begin{array}{ccc} S^{(1)}_1 \times S^{(2)}_1 & \cdots & S^{(1)}_1 \times S^{(2)}_{r_2} \\ S^{(1)}_2 \times S^{(2)}_1 & \cdots & S^{(1)}_2 \times S^{(2)}_{r_2} \\ \vdots & \ddots & \vdots \\ S^{(1)}_{r_1} \times S^{(2)}_1 & \cdots & S^{(1)}_{r_1} \times S^{(2)}_{r_2} \end{array} \right\}.$$

Therefore, (ii) follows. □

In the particular case of the Petersen graph, P, and the 3-cube graph, Q_3, we have $a^d_{-2}(P \times Q_3) = 4 = a^d_{-1}(P)a^d_{-1}(Q_3)$, $\psi^d_{-2}(P \times Q_3) = 20 = \psi^d_{-1}(P)\psi^d_{-1}(Q_3)$ and $16 = a^d_2(P \times Q_3) < a^d_1(P)a^d_1(Q_3) = 20$, $5 = \psi^d_2(P \times Q_3) > \psi^d_1(P)\psi^d_1(Q_3) = 4$.

An example where we cannot apply Theorem 3.1 (i) is the book graph $\Gamma_1 \times \Gamma_2 = K_{1,4} \times K_2$, for $k_1 = 2$ and $k_2 = 0$; the star graph $\Gamma_1 = K_{1,4}$ does not contain defensive 2-alliances, although $\Gamma_1 \times \Gamma_2$ contains some of them and $a^d_2(\Gamma_1 \times \Gamma_2) = 8$.

We note that from Theorem 3.1 we obtain $a^d_{2k}(\Gamma_1 \times \Gamma_2) \leq a^d_k(\Gamma_1)a^d_k(\Gamma_2)$ and $\psi^d_{2k}(\Gamma_1 \times \Gamma_2) \geq \psi^d_k(\Gamma_1)\psi^d_k(\Gamma_2)$. Another interesting consequence of Theorem 3.1 is the following

Corollary 3.2 *Let Γ_1 and Γ_2 be two graphs of order n_1 and n_2 and maximum degree Δ_1 and Δ_2, respectively. Let $s \in \mathbb{Z}$ such that* $\max\{\Delta_1, \Delta_2\} \leq s \leq \Delta_1 + \Delta_2 + k$. *Then*

(i) $a^d_{k-s}(\Gamma_1 \times \Gamma_2) \leq \min\{a^d_k(\Gamma_1), a^d_k(\Gamma_2)\}$,

(ii) $\psi^d_{k-s}(\Gamma_1 \times \Gamma_2) \geq \max\{n_2\psi^d_k(\Gamma_1), n_1\psi^d_k(\Gamma_2)\}$.

As example of equalities we take $\Gamma_1 = P$, $\Gamma_2 = Q_3$, $k = 1$ and $s = 3$. In such a case, $4 = a^d_{-2}(P \times Q_3) = \min\{a^d_1(P), a^d_1(Q_3)\} = \min\{5, 4\}$ and $20 = \psi^d_{-2}(P \times Q_3) = \max\{8\psi^d_1(P), 10\psi^d_1(Q_3)\} = \max\{16, 20\}$.

3.2 Partitioning $\Gamma_1 \times \Gamma_2$ into Global Defensive k-Alliances

Theorem 3.3 *Let $\Pi_{r_i}(\Gamma_i)$ be a partition of a graph Γ_i, of order n_i, into $r_i \geq 1$ global defensive k_i-alliances, $i \in \{1, 2\}$, $r_1 \leq r_2$. Let $x_i = \min_{X \in \Pi_{r_i}(\Gamma_i)}\{|X|\}$. Then,*

(i) $\gamma^d_{k_1+k_2}(\Gamma_1 \times \Gamma_2) \leq \min\{x_1 n_2, x_2 n_1\}$,

(ii) $\psi^{gd}_{k_1+k_2}(\Gamma_1 \times \Gamma_2) \geq \max\{\psi^{gd}_{k_1}(\Gamma_1), \psi^{gd}_{k_2}(\Gamma_2)\}$.

Proof From the procedure shown in the proof of Theorem 3.1 we obtain that for every $S^{(1)}_j \in \Pi_{r_1}(\Gamma_1)$ and every $S^{(2)}_l \in \Pi_{r_2}(\Gamma_2)$, the sets $M_j = S^{(1)}_j \times V_2$ and $N_l = V_1 \times S^{(2)}_l$ are defensive $(k_1 + k_2)$-alliances in $\Gamma_1 \times \Gamma_2$. Moreover M_j and N_l are dominating sets in $\Gamma_1 \times \Gamma_2$. Thus, by taking $S^{(1)}_j$ and $S^{(2)}_l$ of cardinality x_1 and x_2, respectively, we obtain $|M_j| = x_1 n_2$ and $|N_l| = x_2 n_1$, so (i) follows. Moreover, as $\{M_1, \ldots, M_{r_1}\}$ and $\{N_1, \ldots, N_{r_2}\}$ are partitions of $\Gamma_1 \times \Gamma_2$ into global defensive $(k_1 + k_2)$-alliances, (ii) follows. □

Corollary 3.4 *If Γ_i is a graph of order n_i such that $\psi^{gd}_{k_i}(\Gamma_i) \geq 1$, $i \in \{1, 2\}$, then*

$$\gamma^d_{k_1+k_2}(\Gamma_1 \times \Gamma_2) \leq \frac{n_1 n_2}{\max_{i \in \{1,2\}}\{\psi^{gd}_{k_i}(\Gamma_i)\}}.$$

Theorem 3.5 *If Γ_1 contains a global defensive k_1-alliance, then for every $k_2 \in \{-\Delta_2, \ldots, \delta_2\}$, $\Gamma_1 \times \Gamma_2$ contains a global defensive $(k_1 + k_2)$-alliance and $\gamma^d_{k_1+k_2}(\Gamma_1 \times \Gamma_2) \leq \gamma^d_{k_1}(\Gamma_1) n_2$.*

Proof Following a similar procedure used in the proof of Theorem 3.3 (i) we deduce the result. □

For the graph $\Gamma_1 \times \Gamma_2 = C_4 \times Q_3$, by taking $k_1 = 0$ and $k_2 = 1$, we obtain equalities in Theorem 3.3, Corollary 3.4 and Theorem 3.5.

References

[1] Kristiansen, P., Hedetniemi, S. M., Hedetniemi, S. T.: Alliances in graphs. *Journal of Combinatorial Mathematics and Combinatorial Computing*, **48**, 157–177 (2004)

[2] Brigham, R. C., Dutton, R. D., Hedetniemi, S. T.: A sharp lower bound on the powerful alliance number of $C_m \times C_n$. *Congressus Numerantium*, **167**, 57–63 (2004)

[3] Brigham, R. C., Dutton, R. D., Haynes, T. W., et al.: Powerful alliances in graphs. *Discrete Mathematics*, **309**(8), 2140–2147 (2009)

[4] Chellali, M., Haynes, T.: Global alliances and independence in trees. *Discussiones Mathematicae Graph Theory*, **27**(1), 19–27 (2007)

[5] Favaron, O., Fricke, G., Goddard, W., et al.: Offensive alliances in graphs. *Discussiones Mathematicae Graph Theory*, **24**(2), 263–275 (2004)

[6] Fernau, H., Rodríguez, J. A., Sigarreta, J. M.: Offensive k-alliances in graphs. *Discrete Applied Mathematics*, **157**(1), 177–182 (2009)

[7] Haynes, T. W., Hedetniemi, S. T., Henning, M. A.: Global defensive alliances in graphs. *Electronic Journal of Combinatorics*, **10**, 139–146 (2003)

[8] Rodríguez, J. A., Sigarreta, J. M.: Spectral study of alliances in graphs. *Discussiones Mathematicae Graph Theory*, **27**(1), 143–157 (2007)

[9] Rodríguez-Velázquez, J. A., Sigarreta, J. M.: Global alliances in planar graphs. *AKCE International Journal of Graphs and Combinatorics*, **4**(1), 83–98 (2007)

[10] Rodríguez-Velázquez, J. A., Yero, I. G., Sigarreta, J. M.: Defensive k-alliances in graphs. *Applied Mathematics Letters*, **22**, 96–100 (2009)

[11] Rodríguez-Velázquez, J. A., Sigarreta, J. M.: Global defensive k-alliances in graphs. *Discrete Applied Mathematics*, **157**, 211–218 (2009)

[12] Shafique, K. H., Dutton, R. D.: Maximum alliance-free and minimum alliance-cover sets. *Congressus Numerantium*, **162**, 139–146 (2003)

[13] Shafique, K. H., Dutton, R.: A tight bound on the cardinalities of maximun alliance-free and minimun alliance-cover sets. *Journal of Combinatorial Mathematics and Combinatorial Compututing*, **56**, 139–145 (2006)

[14] Shafique, K. H.: Partitioning a Graph in Alliances and its Application to Data Clustering, Ph. D. Thesis, School of Computer Science, University of Central Florida Orlando, 2004

[15] Sigarreta, J. M., Bermudo, S., Fernau, H.: On the complement graph and defensive k-alliances. *Discrete Applied Mathematics*, **157**(8), 1687–1695 (2009)

[16] Sigarreta, J. M., Rodríguez, J. A.: On defensive alliance and line graphs. *Applied Mathematics Letters*, **19**(12), 1345–1350 (2006)

[17] Sigarreta, J. M., Rodríguez, J. A.: On the global offensive alliance number of a graph. *Discrete Applied Mathematics*, **157**(2), 219–226 (2009)

[18] Shafique, K. H., Dutton, R. D.: On satisfactory partitioning of graphs. *Congressus Numerantium*, **154**, 183–194 (2002)

[19] Eroh, L., Gera, R.: Global alliance partition in trees. *Journal of Combinatorial Mathematics and Combinatorial Compututing*, **66**, 161–169 (2008)

[20] Eroh, L., Gera, R.: Alliance partition number in graphs. Submitted

[21] Haynes, T. W., Lachniet, J. A.: The alliance partition number of grid graphs. *AKCE International Journal of Graphs and Combinatorics*, **4**(1), 51–59 (2007)

[22] Mohar, B.: Isoperimetric numbers of graphs. *Journal of Combinatorial Theory, Series B*, **47**, 274–291 (1989)

[23] Kahale, N.: Isoperimetric inequalities and eigenvalues. *SIAM Journal on Discrete Mathematics*, **10**(1), 30–40 (1997)

[24] Kwak, J. H., Lee, J., Sohn, M. Y.: Isoperimetric numbers of graph bundles. *Graphs and Combinatorics*, **39**, 19–31 (1996)

[25] Merris, R.: A survey of graph Laplacians. *Linear and Multilinear Algebra*, **39**, 19–31 (1995)