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Alliance free and alliance cover sets

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Abstract A *defensive* (*offensive*) *k-alliance* in $\Gamma = (V, E)$ is a set $S \subseteq V$ such that every v in S (in the boundary of S) has at least k more neighbors in S than it has in $V \setminus S$. A set $X \subseteq V$ is *defensive* (*offensive*) *k-alliance free,* if for all defensive (offensive) k-alliance S, $S \setminus X \neq \emptyset$, i.e., X does not contain any defensive (offensive) k-alliance as a subset. A set $Y \subseteq V$ is a *defensive* (*offensive*) *k-alliance cover*, if for all defensive (offensive) k-alliance S, $S \cap Y \neq \emptyset$, i.e., Y contains at least one vertex from each defensive (offensive) k-alliance of Γ. In this paper we show several mathematical properties of defensive (offensive) k-alliance free sets and defensive (offensive) k-alliance cover sets, including tight bounds on their cardinality.

Keywords Defensive alliance, offensive alliance, alliance free set, alliance cover set

MR(2000) Subject Classification 05C69, 05C70

1 Introduction

In [1], P. Kristiansen, S. M. Hedetniemi and S. T. Hedetniemi introduced several types of alliances in graphs, including defensive and offensive alliances. We are interested in a generalization of alliances, namely k-alliances, given by Shafique and Dutton [2] and studied further

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by several authors [3–9]. In this paper we show several mathematical properties of k-alliance free sets and k-alliance cover sets.

We begin by stating some notation and terminology. In this paper $\Gamma = (V, E)$ denotes a simple graph of order n, size m, minimum degree δ and maximum degree Δ. For a non-empty subset $S \subseteq V$, and any vertex $v \in V$, we denote by $N_S(v)$ the set of neighbors v has in S: $N_S(v) := \{u \in S : u \sim v\}$ and $\delta_S(v) = |N_S(v)|$ denotes the degree of v in S.

The complement of the set S in V is denoted by $\overline{S}$. The boundary of a set $S \subseteq V$ is defined as $\partial S := \cup_{v \in S} N_{\overline{S}}(v)$. A nonempty set of vertices $S \subseteq V$ is called a *defensive* (*offensive*) *k-alliance* in Γ if for every $v \in S$ ($v \in \partial S$), $\delta_S(v) \geq \delta_{\overline{S}}(v) + k$. Hereafter, if there is no restriction on the values of k, we assume that $k \in \{-\Delta, ..., \Delta\}$. Notice that any vertex subset is an offensive k-alliance for $k \in \{-\Delta, 1 - \Delta, 2 - \Delta\}$. A defensive (offensive) k-alliance is *global* if it is a dominating set.

A set $X \subseteq V$ is *defensive* (*offensive*) *k-alliance free,* k-daf (k-oaf), if for all defensive (offensive) k-alliance S, $S \setminus X \neq \emptyset$, i.e., X does not contain any defensive (offensive) k-alliance as a subset [2, 10]. A defensive (offensive) k-alliance free set X is *maximal* if for every defensive (offensive) k-alliance free set Y, $X \not\subset Y$.

A *maximum* k-daf (k-oaf) set is a maximal (k-oaf) k-daf set of largest cardinality.

A set $Y \subseteq V$ is a *defensive* (*offensive*) *k-alliance cover*, k-dac (k-oac), if for all defensive (offensive) k-alliances S, $S \cap Y \neq \emptyset$, i.e., Y contains at least one vertex from each defensive (offensive) k-alliance of Γ. A k-dac (k-oac) set Y is *minimal* if no proper subset of Y is a defensive (offensive) k-alliance cover set. A *minimum* k-dac (k-oac) set is a minimal cover set of smallest cardinality. For short, in the case of a global offensive k-alliance cover (free) set we will write k-goac (k-goaf).

Remark 1.1

(i) If X is a minimal k-dac (k-oac) set then, for all $v \in X$, there exists a defensive (offensive) k-alliance S_v for which $S_v \cap X = \{v\}$.

(ii) If X is a maximal k-daf (k-oaf) set, then, for all $v \in \overline{X}$, there exists $S_v \subseteq X$ such that $S_v \cup \{v\}$ is a defensive (offensive) k-alliance.

Associated with the characteristic sets defined above we have the following invariants:

$a_k(\Gamma)$: minimum cardinality of a defensive k-alliance in Γ.

$\gamma_k(\Gamma)$: minimum cardinality of a global defensive k-alliance in Γ.

$\gamma_k^o(\Gamma)$: minimum cardinality of a global offensive k-alliance in Γ.

$\phi_k(\Gamma)$: cardinality of a maximum k-daf set in Γ.

$\phi_k^o(\Gamma)$: cardinality of a maximum k-oaf set in Γ.

$\phi_k^{go}(\Gamma)$: cardinality of a maximum k-goaf set in Γ.

$\zeta_k(\Gamma)$: cardinality of a minimum k-dac set in Γ.

$\zeta_k^o(\Gamma)$: cardinality of a minimum k-oac set in Γ.

$\zeta_k^{go}(\Gamma)$: cardinality of a minimum k-goac set in Γ.

The following duality between alliance cover and alliance free sets was shown in [2, 10].

Remark 1.2 X is a defensive (offensive) k-alliance cover set if and only if $\overline{X}$ is defensive (offensive) k-alliance free.

Corollary 1.3 $\phi_k(\Gamma) + \zeta_k(\Gamma) = \phi_k^o(\Gamma) + \zeta_k^o(\Gamma) = n$.

2 Alliance cover and alliance free sets

We begin by studying the structure of a set according to the structure of its complementary set.

Theorem 2.1 *If X is a minimal k-dac set, then $\overline{X}$ is a dominating set.*

Proof By Remark 1.2, if X is a minimal k-dac set, then $\overline{X}$ is a maximal k-daf set. Therefore, for all $v \in X$, there exists $X_v \subseteq \overline{X}$ such that $X_v \cup \{v\}$ is a defensive k-alliance. So, for every $u \in X_v$, $\delta_{X_v}(u) + \delta_{\{v\}}(u) = \delta_{X_v \cup \{v\}}(u) \geq \delta_{\overline{X_v \cup \{v\}}}(u) + k = \delta_{\overline{X_v}}(u) - \delta_{\{v\}}(u) + k$. On the other hand, as X_v is not a defensive k-alliance, there exists $w \in X_v$ such that $\delta_{X_v}(w) < \delta_{\overline{X_v}}(w) + k$. Hence, by the above inequalities, $\delta_{\overline{X_v}}(w) + k + \delta_{\{v\}}(w) > \delta_{\overline{X_v}}(w) - \delta_{\{v\}}(w) + k$. Thus, $2\delta_{\{v\}}(w) > 0$ and, as a consequence, v is adjacent to w.

Notice that there exist minimal k-oac sets such that their complement sets are not dominating sets. For instance we consider the graph obtained from the cycle graph C_8 by adding the edge $\{v_1, v_3\}$ and the edge $\{v_5, v_7\}$. In this graph the set $S = \{v_2, v_3, v_5, v_6, v_7\}$ is a minimal 0-oac but $\bar{S}$ is not a dominating set.

Theorem 2.2 *If X is a minimal k-dac set, then $\overline{X}$ is a global offensive k-alliance.*

Proof If $X \subset V$ is a minimal k-dac set, then for every $v \in X$ there exists a defensive k-alliance S_v such that $S_v \cap X = \{v\}$. Hence, $\delta_{S_v}(v) \geq \delta_{\overline{S_v}}(v) + k$ and $\delta_{\overline{X}}(v) \geq \delta_{S_v}(v) \geq \delta_{\overline{S_v}}(v) + k \geq \delta_X(v) + k$. Therefore, for every $v \in X$, we have $\delta_{\overline{X}}(v) \geq \delta_X(v) + k$. On the other hand, by Theorem 2.1, $\overline{X}$ is a dominating set. In consequence, $\overline{X}$ is a global offensive k-alliance in Γ.

Corollary 2.3 $\phi_k(\Gamma) \geq \gamma_k^o(\Gamma)$ *and* $\zeta_k(\Gamma) \leq n - \gamma_k^o(\Gamma)$.

Notice that if one vertex $v \in V$ belongs to any offensive k-alliance, then $V \setminus \{v\}$ is a k-oaf set. Hence, $\delta(v) < k$. So, if $k \leq \delta$ and X is a minimal k-oac set, then $|X| \geq 2$.

Theorem 2.4 *For every $k \in \{2 - \Delta, ..., \Delta\}$, if X is a minimal k-goac set such that $|X| \geq 2$, then $\overline{X}$ is an offensive $(k-2)$-alliance. Moreover, if $k \in \{3, ..., \Delta\}$, then $\overline{X}$ is a global offensive $(k-2)$-alliance.*

Proof If $X \subset V$ is a minimal k-goac set, then for all $v \in X$ there exists a global offensive k-alliance, S_v, such that $S_v \cap X = \{v\}$. Hence, for every $u \in \overline{S_v}$, $1 + \delta_{\overline{X}}(u) \geq \delta_{S_v}(u) \geq \delta_{\overline{S_v}}(u) + k \geq \delta_X(u) + k - 1$. As $X \setminus \{v\} \subset \overline{S_v}$, we have $\delta_{\overline{X}}(u) \geq \delta_X(u) + k - 2$ for every $u \in X \setminus \{v\}$. Now we take a vertex $w \in X \setminus \{v\}$ and by the above procedure, taking the vertex w instead of v, we obtain that $\delta_{\overline{X}}(v) \geq \delta_X(v) + k - 2$. Therefore, $\overline{X}$ is an offensive $(k-2)$-alliance. Moreover, if $k > 2$, $\overline{X}$ is a dominating set. So, in such a case, it is a global offensive $(k-2)$-alliance.

Corollary 2.5 *For every $k \in \{3, ..., \delta\}$,* $\phi_k^{go}(\Gamma) \geq \gamma_{k-2}^o(\Gamma)$ *and* $\zeta_k^{go}(\Gamma) \leq n - \gamma_{k-2}^o(\Gamma)$.

Theorem 2.6 *For every* $k \in \{1-\Delta, ..., \Delta-1\}$,

(i) *if* X *is a global offensive* k*-alliance, then* $\overline{X}$ *is* $(1-k)$*-daf;*

(ii) *if* X *is a defensive* k*-alliance, then* $\overline{X}$ *is* $(1-k)$*-goaf.*

Proof (i) If X is a global offensive k-alliance, then for every $v \in \overline{X}$ we have $\delta_X(v) + 1 - k > \delta_{\overline{X}}(v)$. Hence, the set $\overline{X}$ is not a defensive $(1-k)$-alliance. Moreover, if $Y \subset \overline{X}$, then for every $y \in Y$ we have $\delta_{\overline{Y}}(y) + 1 - k \geq \delta_X(y) + 1 - k > \delta_{\overline{X}}(y) \geq \delta_Y(y)$. Thus, the set Y is not a defensive $(1-k)$-alliance. Therefore, $\overline{X}$ is a $(1-k)$-daf set.

(ii) If X is a defensive k-alliance, then for every $v \in X$ we have $\delta_{\overline{X}}(v) < \delta_X(v) + (1-k)$. So, $\overline{X}$ is not a global offensive $(1-k)$-alliance. Moreover, for every $S \subset \overline{X}$ and $v \in X \subset \overline{S}$ it is satisfied $\delta_S(v) \leq \delta_{\overline{X}}(v) < \delta_X(v) + (1-k) \leq \delta_{\overline{S}}(v) + (1-k)$, in consequence, S is not a global offensive $(1-k)$-alliance.

Corollary 2.7 *For every* $k \in \{1-\Delta, ..., \Delta-1\}$,

(i) $\zeta_{1-k}(\Gamma) \leq \gamma_k^o(\Gamma)$ *and* $\phi_{1-k}(\Gamma) \geq n - \gamma_k^o(\Gamma)$*;*

(ii) $\zeta_{1-k}^{go}(\Gamma) \leq a_k(\Gamma)$.

Notice that all equalities in the above corollaries are attained for the complete graph of order n where $\phi_k(K_n) = n - \zeta_k(K_n) = \gamma_k^o(K_n) = \lceil \frac{n+k-1}{2} \rceil$ and $\zeta_{1-k}^{go}(K_n) = n - \phi_{1-k}^{go}(K_n) = a_k(K_n) = \lceil \frac{n+k+1}{2} \rceil$.

As we show in the following table, by combining some of the above results we can deduce basic properties on alliance free sets and alliance cover sets. For the restrictions on k, see the premises of the corresponding results.

2.1 Table

Rem. 1.2 and Th. 2.1	Any maximal k-daf set is a dominating set.
Rem. 1.2 and Th. 2.2	Any maximal k-daf set is a global offensive k-alliance.
Rem. 1.2 and Th. 2.6	Any global offensive k-alliance is a $(1-k)$-dac set.
Th. 2.2 and Th. 2.6	Any minimal k-dac set is $(1-k)$-daf.
Th. 2.4 and Th. 2.6	Any minimal k-goac set of cardinality at least 2 is $(3-k)$-daf.

2.2 Monotony of $\phi_k^{go}(\Gamma)$ and $\phi_k(\Gamma)$

Theorem 2.8 *If* X *is a* k*-goaf set,* $k \in \{1, ..., \Delta-2\}$*, such that* $|X| \leq n-2$*, then there exists* $v \in \overline{X}$ *such that* $X \cup \{v\}$ *is a* $(k+2)$*-goaf set.*

Proof Let us suppose that for every $x \in \overline{X}$, $X \cup \{x\}$ is not a $(k+2)$-goaf set. Let $v \in \overline{X}$ and let $S_v \subset X$, such that $S_v \cup \{v\}$ is a global offensive $(k+2)$-alliance in Γ. Then for every $u \in \overline{S_v \cup \{v\}} = \overline{S_v} \setminus \{v\}$ we have $\delta_{S_v}(u) = \delta_{S_v \cup \{v\}}(u) - \delta_{\{v\}}(u) \geq \delta_{\overline{S_v \cup \{v\}}}(u) - \delta_{\{v\}}(u) + k + 2 = \delta_{\overline{S_v}}(u) - 2\delta_{\{v\}}(u) + k + 2 \geq \delta_{\overline{S_v}}(u) + k$. So, for every $u \in \overline{X} \setminus \{v\} \subset \overline{S_v} \setminus \{v\}$, $\delta_X(u) \geq \delta_{S_v}(u) \geq \delta_{\overline{S_v}}(u) + k \geq \delta_{\overline{X}}(u) + k$. Now we take a vertex $w \in \overline{X} \setminus \{v\}$ and by the above procedure, taking the vertex w instead of v, we obtain that $\delta_X(v) \geq \delta_{\overline{X}}(v) + k$. So, X is a global offensive k-alliance, a contradiction.

If X is a k-goaf for $k \leq \delta$, then $|X| \leq n-2$, as a consequence, the above result can be simplified as follows.

Corollary 2.9 *If X is a k-goaf set, $k \in \{1, ..., \delta\}$, then there exists $v \in \overline{X}$ such that $X \cup \{v\}$ is a $(k+2)$-goaf set.*

It is easy to check the monotony of ϕ_k^{go}, i.e., $\phi_k^{go}(\Gamma) \leq \phi_{k+1}^{go}(\Gamma)$. As we can see below, Theorem 2.8 leads to an interesting property about the monotony of ϕ_k^{go}.

Corollary 2.10 *For every $k \in \{1, ..., \min\{\delta, \Delta-2\}\}$ and $r \in \{1, ..., \lfloor \frac{\Delta-k}{2} \rfloor\}$,*

$$\phi_k^{go}(\Gamma) + r \leq \phi_{k+2r}^{go}(\Gamma).$$

Theorem 2.11 *If X is a k-daf set and $v \in \overline{X}$, then $X \cup \{v\}$ is $(k+2)-daf$.*

Proof Let us suppose that there exists a defensive $(k+2)$-alliance A such that $A \subseteq X \cup \{v\}$. If $v \notin A$, then $A \subset X$, a contradiction because every defensive (k+2)-alliance is a defensive k-alliance. If $v \in A$, let $B = A \setminus \{v\}$. For every $u \in B$ we have, $\delta_B(u) = \delta_A(u) - \delta_{\{v\}}(u) \geq \delta_{\overline{A}}(u) + k + 2 - \delta_{\{v\}}(u) \geq \delta_{\overline{B}}(u) + k + 2\left(1 - \delta_{\{v\}}(u)\right) \geq \delta_{\overline{B}}(u) + k$. So, $B \subseteq X$ is a defensive k-alliance, a contradiction.

Corollary 2.12 *For every $k \in \{-\Delta, ..., \Delta-2\}$ and $r \in \{1, ..., \lfloor \frac{\Delta-k}{2} \rfloor\}$,*

$$\phi_k(\Gamma) + r \leq \phi_{k+2r}(\Gamma).$$

3 Tight bounds

A dominating set $S \subset V$ is a *global boundary offensive k-alliance* if for every $v \in \overline{S}$, $\delta_S(v) = \delta_{\overline{S}}(v) + k$ [8].

Lemma 3.1 *If $\{X, Y\}$ is a vertex partition of a graph Γ into two global boundary offensive 0-alliances, then X and Y are minimal global offensive 0-alliances in Γ.*

Proof Let us suppose, for instance, that X is not a minimal global offensive 0-alliances, then, there exists $A \subset X$, such that, $X \setminus A \neq \emptyset$ and A is a global offensive 0-alliance. Thus, for every $v \in \overline{A}$, $\delta_X(v) \geq \delta_A(v) \geq \delta_{\overline{A}}(v) \geq \delta_Y(v)$.

As $Y \subset \overline{A}$ and $\{X, Y\}$ is a vertex partition of the graph into two global boundary offensive 0-alliances, then for every $v \in Y$, $\delta_Y(v) = \delta_X(v) \geq \delta_A(v) \geq \delta_{\overline{A}}(v) \geq \delta_Y(v)$. Thus, for every $v \in Y$ we have $\delta_A(v) = \delta_{\overline{A}}(v) = \delta_Y(v) + \delta_{X\setminus A}(v) = \delta_X(v) + \delta_{X\setminus A}(v) = \delta_A(v) + 2\delta_{X\setminus A}(v)$. Hence, we have that Y is a dominating set and for every $v \in Y$, $\delta_{X\setminus A}(v) = 0$, a contradiction. So, X and Y are minimal global offensive 0-alliances.

Theorem 3.2 *For every $k \in \{0, ..., \Delta\}$, $\phi_k^{go}(\Gamma) \geq \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{k}{2} \right\rfloor - 1$.*

Proof First, we will prove the case $k = 0$. Let $\{X, Y\}$ be a partition of the vertex set, such that $|X| = \lfloor \frac{n}{2} \rfloor$, $|Y| = \lceil \frac{n}{2} \rceil$ and there is a minimum number of edges between X and Y. If X (or Y) is a 0-goaf set, then $\phi_0^{go}(\Gamma) \geq \lfloor \frac{n}{2} \rfloor - 1$. We suppose there exist $A \subseteq X$ and $B \subseteq Y$, such that A and B are global offensive 0-alliances. Hence $\delta_X(v) \geq \delta_A(v) \geq \delta_{\bar{A}}(v) \geq \delta_Y(v)$, $\forall v \in \bar{A}$, and $\delta_Y(v) \geq \delta_B(v) \geq \delta_{\bar{B}}(v) \geq \delta_X(v)$, $\forall v \in \bar{B}$. As $Y \subset \bar{A}$ and $X \subset \bar{B}$ we have, for every $v \in Y$, $\delta_X(v) \geq \delta_Y(v)$ and for every $v \in X$, $\delta_Y(v) \geq \delta_X(v)$.

For any $y \in Y$ and $x \in X$, let us take $X' = X \setminus \{x\} \cup \{y\}$ and $Y' = Y \setminus \{y\} \cup \{x\}$. If $\delta_X(y) > \delta_Y(y)$ or $\delta_Y(x) > \delta_X(x)$ then, the edge cutset between X' and Y' is lesser than the other one between X and Y, a contradiction. Therefore $\delta_X(y) = \delta_Y(y)$ and $\delta_Y(x) = \delta_X(x)$ and, as a consequence, $\{X, Y\}$ is a partition of the vertex set into two global boundary offensive 0-alliances. Now, by using Lemma 3.1 we obtain that X and Y are minimal global offensive 0-alliances. As a consequence, $\phi_0^{go}(\Gamma) \geq \lfloor \frac{n}{2} \rfloor - 1$.

Now, let us prove the case $k > 0$. Case 1: $\phi_k^{go}(\Gamma) \geq n-2$. Since $n-1 \geq \lfloor \frac{2n-1}{2} \rfloor \geq \lfloor \frac{n+\Delta}{2} \rfloor \geq \lfloor \frac{n+k}{2} \rfloor \geq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{k}{2} \rfloor$, we have $\phi_k^{go}(\Gamma) \geq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{k}{2} \rfloor - 1$. Case 2: $\phi_k^{go}(\Gamma) < n-2$. As every k-goaf set is also a $(k+1)$-goaf set, $\phi_1^{go}(\Gamma) \geq \phi_0^{go}(\Gamma) \geq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{1}{2} \rfloor - 1$, then the statement is true for $k = 1$. Hence, we will proceed by induction on k. Let us assume that the statement is true for an arbitrary $k \in \{2, ..., \Delta - 2\}$, that is, there exists a maximal k-goaf set X in Γ such that, $|X| = \phi_k^{go}(\Gamma) \geq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{k}{2} \rfloor - 1$. Now, by Theorem 2.8, there exists $v \in \overline{X}$, such that $X \cup \{v\}$ is a $(k+2)$-goaf set. Therefore, $\phi_{k+2}^{go}(\Gamma) \geq |X \cup \{v\}| \geq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{k}{2} \rfloor = \lfloor \frac{n}{2} \rfloor + \lfloor \frac{k+2}{2} \rfloor - 1$. So, the proof is complete.

The above bound is attained, for instance, in the case of the complete graph if n and k are both even or if n and k have different parity: $\phi_k^{go}(K_n) = \left\lfloor \frac{n+k-2}{2} \right\rfloor$.

Theorem 3.3 $\left\lceil \frac{\delta + k - 2}{2} \right\rceil \leq \phi_k^o(\Gamma) \leq \left\lfloor \frac{2n - \delta + k - 3}{2} \right\rfloor$.

Proof If X is a k-oaf set, then $\delta_X(v)+1 \leq \delta_{\overline{X}}(v)+k$, for some $v \in \partial X$. Therefore, $\delta(v)+1-k = \delta_X(v) + \delta_{\overline{X}}(v) + 1 - k \leq 2\delta_{\overline{X}}(v) \leq 2(n - |X| - 1)$. Thus, the upper bound is deduced.

If X is a maximal k-oaf set, then $\overline{X}$ is a minimal k-oac set. Thus, for all $v \in \overline{X}$, there exists an offensive k-alliance S_v such that $S_v \cap \overline{X} = \{v\}$. Hence, $\delta_{S_v}(u) \geq \delta_{\overline{S_v}}(u) + k$, for every $u \in \partial S_v$. Therefore, $\delta(u) + k \leq 2\delta_{S_v}(u) \leq 2|S_v| \leq 2(|X| + 1)$. Thus, the lower bound follows.

The above bounds are attained, for instance, for the complete graph: $\phi_k^o(K_n) = \left\lceil \frac{n+k-3}{2} \right\rceil$.

For every $k \in \{0, ..., \Delta\}$ it was established in [10] that $\phi_k(\Gamma) \geq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{k}{2} \rfloor$. The next result shows other bounds on $\phi_k(\Gamma)$.

Theorem 3.4 *For any connected graph* Γ*,*

$$\left\lceil \frac{n(k + \mu) - \mu}{n + \mu} \right\rceil \leq \phi_k(\Gamma) \leq \left\lfloor \frac{2n + k - \delta - 1}{2} \right\rfloor,$$

where μ *denotes the algebraic connectivity of* Γ*.*

Proof It was shown in [4] that the defensive k-alliance number is bounded by $a_k(\Gamma) \geq \left\lceil \frac{n(\mu+k+1)}{n+\mu} \right\rceil$. On the other hand, if S is a defensive k-alliance of cardinality $a_k(\Gamma)$, then for all $v \in S$ we have that $S \setminus \{v\}$ is a k-daf set. Thus, $\phi_k(\Gamma) \geq a_k(\Gamma) - 1$. Hence, the lower bound on $\phi_k(\Gamma)$ follows.

Moreover, if X is a k-daf set, then $\delta_X(v) + 1 \leq \delta_{\overline{X}}(v) + k$, for some $v \in X$. Therefore, $\delta(v) + 1 - k = \delta_X(v) + \delta_{\overline{X}}(v) + 1 - k \leq 2\delta_{\overline{X}}(v) \leq 2(n - |X|)$. Thus, the upper bound follows.

The above bound is sharp as we can check, for instance, for the complete graph $\Gamma = K_n$. As the algebraic connectivity of K_n is $\mu = n$, the above theorem gives the exact value of $\phi_k(K_n) = \left\lceil \frac{n+k-1}{2} \right\rceil$.

Theorem 3.5 *For any connected graph* Γ, $\zeta_k(\Gamma) \leq \frac{n}{\mu_*}\left(\mu_* - \left\lceil \frac{\delta + k}{2} \right\rceil\right)$, *where* μ_* *denotes the Laplacian spectral radius of* Γ*.*

Proof The result immediately follows from Corollary 2.3 and the following bound obtained in [3]: $\gamma_k^o(\Gamma) \geq \frac{n}{\mu_*}\left\lceil\frac{\delta+k}{2}\right\rceil$.

The above bound is tight. For instance, we consider the complete graph $\Gamma = K_n$ for which the Laplacian spectral radius is $\mu_* = n$. In such a case, the above theorem gives the exact value $\zeta_k(K_n) = \left\lceil\frac{n-k}{2}\right\rceil$.

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