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On global offensive k-alliances in graphs

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ABSTRACT

We investigate the relationship between global offensive k-alliances and some characteristic sets of a graph including r-dependent sets, τ-dominating sets and standard dominating sets. In addition, we discuss the close relationships that exist among the (global) offensive k_i-alliance number of Γ_i, $i \in \{1, 2\}$, and the (global) offensive k-alliance number of $\Gamma_1 \times \Gamma_2$, for some specific values of k. As a consequence of the study, we obtain bounds on the global offensive k-alliance number in terms of several parameters of the graph.

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1. Introduction

The mathematical properties of alliances in graphs were first studied by Kristiansen et al. [1]. They proposed alliances of different types that have been extensively studied during the last four years. These alliance types are called *defensive alliances* [1–6], *offensive alliances* [7–9] and *dual alliances* or *powerful alliances* [10]. A generalization of these alliances called k-alliances was presented by Shafique and Dutton [11]. We are interested in the study of the mathematical properties of global offensive k-alliances.

We begin by stating the terminology used. Throughout this article, $\Gamma = (V, E)$ denotes a simple graph of order $|V| = n$. We denote two adjacent vertices u and v by $u \sim v$. For a nonempty set $S \subseteq V$, and a vertex $v \in V$, $N_S(v)$ denotes the set of neighbors that v has in S: $N_S(v) := \{u \in S : u \sim v\}$, and the degree of v in S will be denoted by $\delta_S(v) = |N_S(v)|$. We denote the degree of a vertex $v \in V$ by $\delta(v)$, the minimum degree of Γ by δ and the maximum degree by Δ. The complement of the set S in V is denoted by $\overline{S}$ and the boundary of S is defined by $\partial(S) := \bigcup_{v\in S} N_{\overline{S}}(v)$.

A set $S \subseteq V$ is a *dominating set* in Γ if for every vertex $v \in \overline{S}$, $\delta_S(v) > 0$ (every vertex in $\overline{S}$ is adjacent to at least one vertex in S). The *domination number* of Γ, denoted by $\gamma(\Gamma)$, is the minimum cardinality of a dominating set in Γ. For $k \in \{2 - \Delta, \dots, \Delta\}$, a nonempty set $S \subseteq V$ is an *offensive k-alliance* in Γ if

$$\delta_S(v) \geq \delta_{\overline{S}}(v) + k, \quad \forall v \in \partial(S) \tag{1}$$

or, equivalently,

$$\delta(v) \geq 2\delta_{\overline{S}}(v) + k, \quad \forall v \in \partial(S). \tag{2}$$

It is clear that if $k > \Delta$, no set S satisfies (1) and, if $k < 2 - \Delta$, all the subsets of V satisfy it. An offensive k-alliance S is called *global* if it is a dominating set. The *offensive k-alliance number* of Γ, denoted by $a^o_k(\Gamma)$, is defined as the minimum cardinality

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of an offensive k-alliance in Γ. The *global offensive k-alliance number* of Γ, denoted by $\gamma_k^o(\Gamma)$, is defined as the minimum cardinality of a global offensive k-alliance in Γ. Notice that $\gamma_k^o(\Gamma) \geq a_k^o(\Gamma)$ and $\gamma_{k+1}^o(\Gamma) \geq \gamma_k^o(\Gamma) \geq \gamma(\Gamma)$.

In addition, if every vertex of Γ has even degree and k is odd, $k = 2l - 1$, then every global offensive $(2l-1)$-alliance in Γ is a global offensive $(2l)$-alliance. Hence, in such a case, $\gamma_{2l-1}^o(\Gamma) = \gamma_{2l}^o(\Gamma)$. Analogously, if every vertex of Γ has odd degree and k is even, $k = 2l$, then every global offensive $(2l)$-alliance in Γ is a global offensive $(2l+1)$-alliance. Hence, in such a case, $\gamma_{2l}^o(\Gamma) = \gamma_{2l+1}^o(\Gamma)$.

2. The global offensive k-alliance number for some families of graphs

The problem of finding the global offensive k-alliance number is NP-complete [8]. Even so, for some graphs it is possible to obtain this number. For instance, it is satisfied that for the family of the complete graphs, K_n, of order $n\gamma_k^o(K_n) = \left\lceil \frac{n+k-1}{2} \right\rceil$, for any cycle, C_n, of order n,

$$\gamma_k^o(C_n) = \begin{cases} \left\lceil \dfrac{n}{3} \right\rceil, & \text{for } k = 0, \\ \left\lceil \dfrac{n}{2} \right\rceil, & \text{for } k = 1, 2, \end{cases}$$

and for any path, P_n, of order n,

$$\gamma_k^o(P_n) = \begin{cases} \left\lceil \dfrac{n}{3} \right\rceil, & \text{for } k = 0, \\ \left\lfloor \dfrac{n}{2} \right\rfloor + k - 1, & \text{for } k = 1, 2. \end{cases}$$

Remark 2.1. Let $\Gamma = K_{r,t}$ be a complete bipartite graph with $t \leq r$. For every $k \in \{2 - r, \dots, r\}$,
(a) if $k \geq t + 1$, then $\gamma_k^o(\Gamma) = r$,
(b) if $k \leq t$ and $\left\lceil \frac{r+k}{2} \right\rceil + \left\lceil \frac{t+k}{2} \right\rceil \geq t$, then $\gamma_k^o(\Gamma) = t$,
(c) if $-t < k \leq t$ and $\left\lceil \frac{r+k}{2} \right\rceil + \left\lceil \frac{t+k}{2} \right\rceil < t$, then $\gamma_k^o(\Gamma) = \left\lceil \frac{r+k}{2} \right\rceil + \left\lceil \frac{t+k}{2} \right\rceil$,
(d) if $k \leq -t$ and $\left\lceil \frac{r+k}{2} \right\rceil + \left\lceil \frac{t+k}{2} \right\rceil < t$, then $\gamma_k^o(\Gamma) = \min\{t, 1 + \left\lceil \frac{r+k}{2} \right\rceil\}$.

Proof. (a) Let $\{V_t, V_r\}$ be the bi-partition of the vertex set of Γ. Since V_r is a global offensive k-alliance, we only need to show that for every global offensive k-alliance S, $V_r \subseteq S$. If $v \in \overline{S}$ it satisfies $\delta_S(v) \geq \delta_{\overline{S}}(v) + k > t$; in consequence $v \in V_t$. Therefore, $\overline{S} \subseteq V_t$ or, equivalently, $V_r \subseteq S$. Thus, we conclude that $\gamma_k^o(\Gamma) = r$.

(b) If $k \leq t$, it is clear that V_t is a global offensive k-alliance, then $\gamma_k^o(\Gamma) \leq t$. We suppose that $\left\lceil \frac{r+k}{2} \right\rceil + \left\lceil \frac{t+k}{2} \right\rceil \geq t$ and there exists a global offensive k-alliance $S = A \cup B$ such that $A \subseteq V_r$, $B \subseteq V_t$ and $|S| < t$. In such a case, as S is a dominating set, $B \neq \emptyset$. Since S is a global offensive k-alliance, $2|B| \geq t + k$ and $2|A| \geq r + k$. Then we have $t > |S| \geq \left\lceil \frac{r+k}{2} \right\rceil + \left\lceil \frac{t+k}{2} \right\rceil \geq t$, a contradiction. Therefore, $\gamma_k^o(\Gamma) = t$.

(c) In the proof of (b) we have shown that if there exists a global offensive k-alliance S of cardinality $|S| < t$, then $|S| \geq \left\lceil \frac{r+k}{2} \right\rceil + \left\lceil \frac{t+k}{2} \right\rceil$. Taking $A \subset V_r$ of cardinality $\left\lceil \frac{r+k}{2} \right\rceil$ and $B \subset V_t$ of cardinality $\left\lceil \frac{t+k}{2} \right\rceil$ we obtain a global offensive k-alliance $S = A \cup B$ of cardinality $|S| = \left\lceil \frac{r+k}{2} \right\rceil + \left\lceil \frac{t+k}{2} \right\rceil$.

(d) Finally, if $S = A \cup B$ where $A \subseteq V_r$, $B \subseteq V_t$, $|A| = \left\lceil \frac{r+k}{2} \right\rceil$ and $|B| = 1$, then S is a global offensive k-alliance. Moreover, S is a minimum global offensive k-alliance if and only if $|S| = 1 + \left\lceil \frac{r+k}{2} \right\rceil \leq t$. □

3. Global offensive k-alliances and r-dependent sets

A set $S \subseteq V$ is an *r-dependent set* in Γ if the maximum degree of a vertex in the subgraph $\langle S \rangle$ induced by S is at most r, i.e., $\delta_S(v) \leq r$, $\forall v \in S$. We denote by $\alpha_r(\Gamma)$ the maximum cardinality of an r-dependent set in Γ [12].

Theorem 3.1. *Let Γ be a graph of order n, minimum degree δ and maximum degree Δ.*

(a) *If S is an r-dependent set in Γ, $r \in \left\{0, \dots, \left\lfloor \frac{\delta-1}{2} \right\rfloor\right\}$, then $\overline{S}$ is a global offensive $(\delta - 2r)$-alliance.*
(b) *If S is a global offensive k-alliance in Γ, $k \in \{2 - \Delta, \dots, \Delta\}$, then $\overline{S}$ is a $\left\lfloor \frac{\Delta-k}{2} \right\rfloor$-dependent set.*
(c) *Let Γ be a δ-regular graph ($\delta > 0$). S is an r-dependent set in Γ, $r \in \left\{0, \dots, \left\lfloor \frac{\delta-1}{2} \right\rfloor\right\}$, if and only if $\overline{S}$ is a global offensive $(\delta - 2r)$-alliance.*

Proof. (a) Let S be an r-dependent set in Γ; then $\delta_S(v) \leq r$ for every $v \in S$. Therefore, $\delta_S(v) + \delta \leq 2\delta_S(v) + \delta_{\overline{S}}(v) \leq 2r + \delta_{\overline{S}}(v)$. As a consequence, $\delta_{\overline{S}}(v) \geq \delta_S(v) + \delta - 2r$, for every $v \in S$. That is, $\overline{S}$ is a global offensive $(\delta - 2r)$-alliance in Γ.

(b) If S is a global offensive k-alliance in Γ, then $\delta(v) \geq 2\delta_{\overline{S}}(v) + k$ for every $v \in \overline{S}$. As a consequence, $\delta_{\overline{S}}(v) \leq \frac{\delta(v)-k}{2} \leq \frac{\Delta-k}{2}$ for every $v \in \overline{S}$, that is, $\overline{S}$ is a $\left\lfloor \frac{\Delta-k}{2} \right\rfloor$-dependent set in Γ.

(c) The result follows immediately from (a) and (b). □

Corollary 3.2. *Let Γ be a graph of order n, minimum degree δ and maximum degree Δ.*

- *For every $k \in \{2-\Delta, \ldots, \Delta\}$, $n - \alpha_{\lfloor \frac{\Delta-k}{2} \rfloor}(\Gamma) \leq \gamma_k^o(\Gamma)$.*
- *For every $k \in \{1, \ldots, \delta\}$, $\gamma_k^o(\Gamma) \leq n - \alpha_{\lfloor \frac{\delta-k}{2} \rfloor}(\Gamma)$.*
- *If Γ is a δ-regular graph ($\delta > 0$), for every $k \in \{1, \ldots, \delta\}$, $\gamma_k^o(\Gamma) = n - \alpha_{\lfloor \frac{\delta-k}{2} \rfloor}(\Gamma)$.*

4. Global offensive k-alliances and τ-dominating sets

Let Γ be a graph without isolated vertices. For a given $\tau \in (0, 1]$, a set $S \subseteq V$ is called a τ-*dominating set* in Γ if $\delta_S(v) \geq \tau\delta(v)$ for every $v \in \bar{S}$. We denote by $\gamma_\tau(\Gamma)$ the minimum cardinality of a τ-dominating set in Γ [13].

Theorem 4.1. *Let Γ be a graph of minimum degree $\delta > 0$ and maximum degree Δ.*
(a) *If $0 < \tau \leq \min\{\frac{k+\delta}{2\delta}, \frac{k+\Delta}{2\Delta}\}$, then every global offensive k-alliance in Γ is a τ-dominating set.*
(b) *If $\max\{\frac{k+\delta}{2\delta}, \frac{k+\Delta}{2\Delta}\} \leq \tau \leq 1$, then every τ-dominating set in Γ is a global offensive k-alliance.*

Proof. (a) If S is a global offensive k-alliance in Γ, then $2\delta_S(v) \geq \delta(v)+k$ for every $v \in \bar{S}$. Therefore, if $0 < \tau \leq \min\{\frac{1}{2}, \frac{k+\delta}{2\delta}\}$, then $\delta_S(v) \geq \frac{\delta(v)+k}{2} \geq \frac{\delta(v)+\delta(2\tau-1)}{2} \geq \tau\delta(v)$. Moreover, if $\frac{1}{2} \leq \tau \leq \frac{k+\Delta}{2\Delta}$, then $\delta_S(v) \geq \frac{\delta(v)+k}{2} \geq \frac{\delta(v)+\Delta(2\tau-1)}{2} \geq \tau\delta(v)$.

(b) Since $\delta > 0$, it is clear that every τ-dominating set is a dominating set. If $\tau \geq \frac{1}{2}$, then $\delta(2\tau-1) \leq \delta(v)(2\tau-1)$, for every vertex v in Γ. Hence, if S is a τ-dominating set and $\frac{k+\delta}{2\delta} \leq \tau$, we have $k \leq (2\tau-1)\delta(v) \leq 2\delta_S(v) - \delta(v)$, for every $v \in \bar{S}$. Thus, S is a global offensive k-alliance in Γ.

On the other hand, if $\tau \leq \frac{1}{2}$, then $\Delta(2\tau-1) \leq \delta(v)(2\tau-1)$, for every vertex v in Γ. Hence, if S is a τ-dominating set and $\frac{k+\Delta}{2\Delta} \leq \tau$, we have $k \leq (2\tau-1)\delta(v) \leq 2\delta_S(v) - \delta(v)$, for every $v \in \bar{S}$. Thus, S is a global offensive k-alliance in Γ. □

Corollary 4.2. *S is a global offensive (0)-alliance in Γ if and only if S is a $(\frac{1}{2})$-dominating set.*

Corollary 4.3. *S is a global offensive k-alliance in a δ-regular graph Γ if and only if S is a $(\frac{k+\delta}{2\delta})$-dominating set in Γ.*

Theorem 4.4. *Let Γ be a graph of order n, minimum degree $\delta > 0$ and maximum degree $\Delta \geq 2$. For every $j \in \{2-\Delta, \ldots, 0\}$ and $k \leq -\frac{j\delta}{\Delta}$ it is satisfied that $\gamma_k^o(\Gamma) + \gamma_j^o(\Gamma) \leq n$.*

Proof. If $j \in \{2-\Delta, \ldots, 0\}$, then there exists $\tau \in \left[\frac{1}{\Delta}, \frac{1}{2}\right]$ such that $j = \Delta(2\tau-1)$. Therefore, if S is a τ-dominating set, then (by Theorem 4.1(b)) S is a global offensive j-alliance. In consequence, $\gamma_j^o(\Gamma) \leq \gamma_\tau(\Gamma)$. Moreover, if $k \leq -\frac{j\delta}{\Delta} = \delta(1-2\tau)$, then $1-\tau \geq \max\{\frac{1}{2}, \frac{k+\delta}{2\delta}\}$. Hence, by Theorem 4.1(b), we have that every $(1-\tau)$-dominating set is a global offensive k-alliance. Thus, $\gamma_k^o(\Gamma) \leq \gamma_{1-\tau}(\Gamma)$. Using that $\gamma_\tau(\Gamma) + \gamma_{1-\tau}(\Gamma) \leq n$ for $0 < \tau < 1$ (see Theorem 9 [13]), we obtain the required result. □

Notice that from Theorem 4.4 we have the following result.

Corollary 4.5. *If Γ is a graph of order n and minimum degree $\delta > 0$, then $\gamma_0^o(\Gamma) \leq \frac{n}{2}$.*

5. Global offensive k-alliances and standard dominating sets

We say that a global offensive k-alliance S is *minimal* if no proper subset $S' \subset S$ is a global offensive k-alliance.

Theorem 5.1. *Let Γ be a graph without isolated vertices and $k \leq 1$. If S is a minimal global offensive k-alliance in Γ, then $\bar{S}$ is a dominating set in Γ.*

Proof. We suppose that there exists $u \in S$ such that $\delta_{\bar{S}}(u) = 0$ and let $S' = S \setminus \{u\}$. Since S is a minimal global offensive k-alliance, and Γ has no isolated vertices, there exists $v \in \overline{S'}$ such that $\delta_{S'}(v) < \delta_{\overline{S'}}(v) + k$. If $v = u$, we have $\delta_S(u) = \delta_{S'}(u) < \delta_{\overline{S'}}(u) + k = k$, a contradiction. If $v \neq u$, we have $\delta_S(v) = \delta_{S'}(v) < \delta_{\overline{S'}}(v) + k = \delta_{\bar{S}}(v) + k$, which is a contradiction too. Thus, $\delta_{\bar{S}}(u) > 0$ for every $u \in S$. □

In the following result $\bar{\Gamma} = (V, \bar{E})$ denotes the complement of $\Gamma = (V, E)$.

Lemma 5.2. *Let Γ be a graph of order n. A dominating set S in $\bar{\Gamma}$ is a global offensive k-alliance in $\bar{\Gamma}$ if and only if $\delta_S(v) - \delta_{\bar{S}}(v) + n + k - 1 \leq 2|S|$ for every $v \in \bar{S}$.*

Proof. We know that a dominating set S in $\bar{\Gamma}$ is a global offensive k-alliance in $\bar{\Gamma}$ if and only if $\bar{\delta}_S(v) \geq \bar{\delta}_{\bar{S}}(v) + k$ for every $v \in \bar{S}$, where $\bar{\delta}_S(v)$ and $\bar{\delta}_{\bar{S}}(v)$ denote the number of vertices that v has in S and $\bar{S}$, respectively, in $\bar{\Gamma}$. Now, using that $\bar{\delta}_S(v) = |S| - \delta_S(v)$ and $\bar{\delta}_{\bar{S}}(v) = |\bar{S}| - 1 - \delta_{\bar{S}}(v) = n - |S| - 1 - \delta_{\bar{S}}(v)$, we get that S is a global offensive k-alliance in $\bar{\Gamma}$ if and only if $|S| - \delta_S(v) \geq n - |S| - 1 + k - \delta_{\bar{S}}(v)$ or, equivalently, if $\delta_S(v) - \delta_{\bar{S}}(v) + n + k - 1 \leq 2|S|$ for every $v \in \bar{S}$. □

Theorem 5.3. *Let Γ be a graph of order n, minimum degree δ and maximum degree Δ.*
(a) *Every dominating set in $\bar{\Gamma} = (V, \bar{E})$, $S \subseteq V$, of cardinality $|S| \geq \left\lceil \frac{n+k+\Delta-1}{2} \right\rceil$ is a global offensive k-alliance in $\bar{\Gamma}$.*
(b) *Every dominating set in $\Gamma = (V, E)$, $S \subseteq V$, of cardinality $|S| \geq \left\lceil \frac{2n+k-\delta-2}{2} \right\rceil$ is a global offensive k-alliance in Γ.*

Proof. If S is a dominating set in $\bar{\Gamma}$ and it satisfies $|S| \geq \left\lceil \frac{n+k+\Delta-1}{2} \right\rceil$, then

$$|S| \geq \frac{n+k+\Delta-1}{2} \geq \frac{\delta_S(v) - \delta_{\bar{S}}(v) + n + k - 1}{2}$$

for every vertex v. Therefore, by Lemma 5.2 we have that S is a global offensive k-alliance in $\bar{\Gamma}$. Thus, the result (a) follows.
Analogously, on replacing Γ by $\bar{\Gamma}$ and taking into account that the maximum degree in $\bar{\Gamma}$ is $n - 1 - \delta$, the result (b) follows. □

6. The Cartesian product of *k*-alliances

In this section we discuss the close relationships that exist among the (global) offensive k_i-alliance number of Γ_i, $i \in \{1, 2\}$, and the (global) offensive k-alliance number of $\Gamma_1 \times \Gamma_2$, for some specific values of k.

Theorem 6.1. *Let $\Gamma_i = (V_i, E_i)$ be a graph of minimum degree δ_i and maximum degree Δ_i, $i \in \{1, 2\}$.*
(a) *If S_i is an offensive k_i-alliance in Γ_i, $i \in \{1, 2\}$, then, for $k = \min\{k_2 - \Delta_1, k_1 - \Delta_2\}$, $S_1 \times S_2$ is an offensive k-alliance in $\Gamma_1 \times \Gamma_2$.*
(b) *Let $S_i \subset V_i$, $i \in \{1, 2\}$. If $S_1 \times S_2$ is an offensive k-alliance in $\Gamma_1 \times \Gamma_2$, then S_1 is an offensive $(k + \delta_2)$-alliance in Γ_1 and S_2 is an offensive $(k + \delta_1)$-alliance in Γ_2; moreover, $k \leq \min\{\Delta_1 - \delta_2, \Delta_2 - \delta_1\}$.*

Proof. If $X = S_1 \times S_2$, then $(u, v) \in \partial X$ if and only if either $u \in \partial S_1$ and $v \in S_2$ or $u \in S_1$ and $v \in \partial S_2$. We distinguish two cases:
Case 1: If $u \in \partial S_1$ and $v \in S_2$, then $\delta_X(u, v) = \delta_{S_1}(u)$ and $\delta_{\overline{X}}(u, v) = \delta_{\overline{S_1}}(u) + \delta(v)$.
Case 2: If $u \in S_1$ and $v \in \partial S_2$, then $\delta_X(u, v) = \delta_{S_2}(v)$ and $\delta_{\overline{X}}(u, v) = \delta(u) + \delta_{\overline{S_2}}(v)$.
(a) In Case 1 we have $\delta_X(u, v) = \delta_{S_1}(u) \geq \delta_{\overline{S_1}}(u) + k_1 = \delta_{\overline{X}}(u, v) - \delta(v) + k_1 \geq \delta_{\overline{X}}(u, v) - \Delta_2 + k_1$ and in Case 2 we obtain $\delta_X(u, v) = \delta_{S_2}(v) \geq \delta_{\overline{S_2}}(v) + k_2 = \delta_{\overline{X}}(u, v) - \delta(u) + k_2 \geq \delta_{\overline{X}}(u, v) - \Delta_1 + k_2$. Hence, for every $(u, v) \in \partial X$, $\delta_X(u, v) \geq \delta_{\overline{X}}(u, v) + k$, with $k = \min\{k_2 - \Delta_1, k_1 - \Delta_2\}$. So, the result follows.
(b) In Case 1 we have $\delta_{S_1}(u) = \delta_X(u, v) \geq \delta_{\overline{X}}(u, v) + k = \delta_{\overline{S_1}}(u) + \delta(v) + k \geq \delta_{\overline{S_1}}(u) + \delta_2 + k$ and in Case 2 we deduce $\delta_{S_2}(v) = \delta_X(u, v) \geq \delta_{\overline{X}}(u, v) + k = \delta_{\overline{S_2}}(v) + \delta(u) + k \geq \delta_{\overline{S_2}}(v) + \delta_1 + k$. Hence, for every $u \in \partial S_1$, $\delta_{S_1}(u) \geq \delta_{\overline{S_1}}(u) + \delta_2 + k$ and for every $v \in \partial S_2$, $\delta_{S_2}(v) \geq \delta_{\overline{S_2}}(v) + \delta_1 + k$. So, the result follows. □

Corollary 6.2. *Let Γ_i be a graph of maximum degree Δ_i, $i \in \{1, 2\}$. Then for every $k \leq \min\{k_1 - \Delta_2, k_2 - \Delta_1\}$, $a_k^o(\Gamma_1 \times \Gamma_2) \leq a_{k_1}^o(\Gamma_1) a_{k_2}^o(\Gamma_2)$.*

For the particular case of the graph $C_4 \times K_4$, we have $a_{-3}^o(C_4 \times K_4) = 2 = a_0^o(C_4) a_1^o(K_4)$.

Theorem 6.3. *Let $\Gamma_2 = (V_2, E_2)$ be a graph of maximum degree Δ_2 and minimum degree δ_2.*
(i) *If S is a global offensive k-alliance in Γ_1, then $S \times V_2$ is a global offensive $(k - \Delta_2)$-alliance in $\Gamma_1 \times \Gamma_2$.*
(ii) *If $S \times V_2$ is a global offensive k-alliance in $\Gamma_1 \times \Gamma_2$, then S is a global offensive $(k + \delta_2)$-alliance in Γ_1; moreover, $k \leq \Delta_1 - \delta_2$, where Δ_1 denotes the maximum degree of Γ_1.*

Proof. (i) We first note that, as S is a dominating set in Γ_1, $X = S \times V_2$ is a dominating set in $\Gamma_1 \times \Gamma_2$. In addition, for every $x_{ij} = (u_i, v_j) \in \bar{X}$ we have $\delta_X(x_{ij}) = \delta_S(u_i)$ and $\delta_{\bar{S}}(u_i) + \Delta_2 \geq \delta_{\bar{S}}(u_i) + \delta(v_j) = \delta_{\bar{X}}(x_{ij})$, so $\delta_X(x_{ij}) = \delta_S(u_i) \geq \delta_{\bar{S}}(u_i) + k \geq \delta_{\bar{X}}(x_{ij}) - \Delta_2 + k$. Thus, X is a global offensive $(k - \Delta_2)$-alliance in $\Gamma_1 \times \Gamma_2$.
(ii) From Theorem 6.1(b) we obtain that S is an offensive $(k + \delta_2)$-alliance in Γ_1 and $k \leq \Delta_1 - \delta_2$. We only need to show that S is a dominating set. As $S \times V_2$ is a dominating set in $\Gamma_1 \times \Gamma_2$, we have that for every $u \in \bar{S}$ and $v \in V_2$ there exists $(a, b) \in S \times V_2$ such that (a, b) is adjacent to (u, v); hence, $b = v$ and a is adjacent to u, so the result follows. □

It is easy to see the following result on domination, $\gamma(\Gamma_1 \times \Gamma_2) \leq n_2 \gamma(\Gamma_1)$, where n_2 is the order of Γ_2. An "analogous" result on global offensive k-alliances can be deduced from Theorem 6.3(i).

Corollary 6.4. *For any graph Γ_1 and any graph Γ_2 of order n_2 and maximum degree Δ_2, $\gamma_{k-\Delta_2}^o(\Gamma_1 \times \Gamma_2) \leq n_2 \gamma_k^o(\Gamma_1)$.*

We emphasize the following particular cases of Corollary 6.4.

Remark 6.5. For any graph Γ,
(a) $\gamma_{k-2}^o(\Gamma \times C_t) \leq t\gamma_k^o(\Gamma)$,
(b) $\gamma_{k-2}^o(\Gamma \times P_t) \leq t\gamma_k^o(\Gamma)$,
(c) $\gamma_{k-t+1}^o(\Gamma \times K_t) \leq t\gamma_k^o(\Gamma)$.

Notice also that if Γ_2 is a regular graph, Theorem 6.3(i) can be simplified as follows.

Corollary 6.6. *Let $\Gamma_2 = (V_2, E_2)$ be a δ-regular graph. A set S is a global offensive k-alliance in Γ_1 if and only if $S \times V_2$ is a global offensive $(k - \delta)$-alliance in $\Gamma_1 \times \Gamma_2$.*

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