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Infinite Chains in the Tree of Numerical Semigroups and Generalizations



MARIANA ROSAS RIBEIRO

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Mariana Rosas Ribeiro

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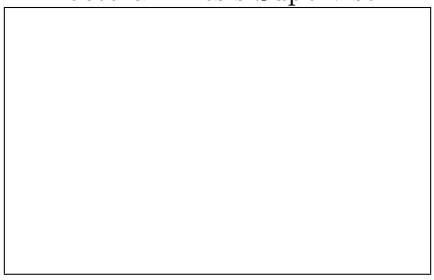
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I STATE that the present study, entitled “Infinite Chains in the Tree of Numerical Semigroups and Generalizations”, presented by Mariana Rosas Ribeiro for the award of the degree of Doctor, has been carried out under my supervision at the Department of Computer Engineering and Mathematics of this university, and that it fulfills all the requirements to be eligible for the International Doctorate Award.

Tarragona, July 9, 2025.

Doctoral Thesis Supervisor



Dr. Maria Bras Amorós

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"The more slowly trees grow at first,
the sounder they are at the core,
and I think the same is true of human beings."
— Henry David Thoreau [1]

Abstract

Numerical semigroups are subsets of the non-negative integers that contain zero, are closed under addition, and have a finite complement in the non-negative integers. Despite their straightforward definition, many fundamental problems have remained unsolved for decades. The overarching theme of this research is the counting of numerical semigroups by genus, a problem initiated by Maria Bras-Amorós, who conjectured that the sequence counting numerical semigroups with a given genus exhibits Fibonacci-like behavior. Significant theoretical and computational advances have been made in understanding this sequence, including bounds and asymptotic properties. The tree of numerical semigroups serves as a fundamental tool in this investigation, as it structures semigroups hierarchically according to their genus.

The central goal of this thesis is the study of infinite chains in the semigroup tree, which correspond to sequences of numerical semigroups with infinitely many descendants. This research refines previous characterizations of infinite chains and demonstrates that, as the genus increases, numerical semigroups belonging to infinite chains become rare. Furthermore, the number of numerical semigroups in infinite chains with multiplicity up to 7 is explicitly computed. Additionally, the thesis extends these concepts to the setting of generalized numerical semigroups (GNS), which are submonoids of $\mathbb{N}_0^d$ with finite complement. A framework for trees of GNSs is explored, and a characterization of infinite chains in this generalized setting is provided.

Resum

Els semigrups numèrics són subconjunts de enters no negatius que contenen el zero, tancats per la suma i tenen complement finit en els enters no negatius. Malgrat la seva senzilla definició, molts problemes fonamentals es mantenen sense resoldre des de fa dècades. El tema global de la meva recerca és el recompte de semigrups numèrics segons el gènere, un problema iniciat per Maria Bras-Amorós, que va conjecturar que la seqüència que compta els semigrups numèrics amb un gènere donat exhibeix un comportament similar al de la seqüència de Fibonacci. S'han fet molts avenços teòrics i computacionals significatius en la comprensió d'aquesta seqüència, incloent-hi fites i propietats asimptòtiques. L'arbre de semigrups numèrics és una eina fonamental en aquesta investigació, ja que estructura els semigrups jeràrquicament segons el seu gènere.

L'objectiu central d'aquesta tesi és l'estudi de les cadenes infinites en l'arbre de semigrups, les quals corresponen a seqüències de semigrups numèrics amb infinits descendents. Aquesta recerca refina caracteritzacions anteriors de cadenes infinites i demostra que, a mesura que el gènere augmenta, els semigrups numèrics que pertanyen a cadenes infinites són cada cop més difícils de trobar. A més, es calcula explícitament el nombre de semigrups numèrics en cadenes infinites amb multiplicitat de fins a 7. Finalment, la tesi amplia aquests conceptes al context dels semigrups numèrics generalitzats (GNS), que són submonoides de $\mathbb{N}_0^d$ amb complement finit. S'explora un marc per als arbres de GNS i es presenta una caracterització de les cadenes infinites en aquest context generalitzat.

Resumen

Los semigrupos numéricos son subconjuntos de enteros no negativos que contienen el cero, cerrados bajo la suma y con un complemento finito en los enteros no negativos. A pesar de su sencilla definición, muchos problemas fundamentales en la teoría han permanecido sin resolver durante décadas. El tema principal de esta investigación es el recuento de semigrupos numéricos por género, un problema iniciado por Maria Bras-Amorós, quien conjeturó que la secuencia que cuenta los semigrupos numéricos con un género dado exhibe un comportamiento similar a la sucesión de Fibonacci. Se han logrado avances teóricos y computacionales significativos en la comprensión de esta secuencia, incluyendo cotas y propiedades asintóticas. El árbol de semigrupos numéricos es una herramienta fundamental en esta investigación, ya que estructura los semigrupos jerárquicamente según su género.

El objetivo central de esta tesis es el estudio de cadenas infinitas en el árbol de semigrupos, las cuales corresponden a secuencias de semigrupos numéricos con infinitos descendientes. Esta investigación refina caracterizaciones previas de cadenas infinitas y demuestra que, a medida que el género aumenta, los semigrupos numéricos que pertenecen a cadenas infinitas se vuelven cada vez más raros. Además, se calcula explícitamente el número de semigrupos numéricos en cadenas infinitas con multiplicidad de hasta 7. Finalmente, la tesis extiende estos conceptos al contexto de los semigrupos numéricos generalizados (GNS), que son submonoidales de $\mathbb{N}_0^d$ con complemento finito. Se explora un marco para los árboles de GNS y se presenta una caracterización de cadenas infinitas en este contexto generalizado.

Resumo

Semigrupos numéricos são subconjuntos dos inteiros não negativos que contêm o zero, são fechados pela adição e possuem complemento finito nos inteiros não negativos. Apesar de sua definição simples, muitos problemas fundamentais na teoria permanecem sem solução há décadas. O tema global desta pesquisa é a contagem de semigrupos numéricos por gênero, um problema iniciado por Maria Bras-Amorós, que conjecturou que a sequência que conta os semigrupos numéricos com um dado gênero exibe um comportamento semelhante ao da sequência de Fibonacci. Avanços teóricos e computacionais significativos vêm sendo feitos na compreensão dessa sequência, incluindo cotas e propriedades assintóticas. A árvore de semigrupos numéricos é uma ferramenta fundamental nesta investigação, à medida que estrutura os semigrupos hierarquicamente de acordo com seu gênero.

O objetivo central desta tese é o estudo de cadeias infinitas na árvore de semigrupos, cadeias essas que correspondem a sequências de semigrupos numéricos com infinitos descendentes. Esta pesquisa refina caracterizações anteriores de cadeias infinitas e demonstra que, à medida que o gênero cresce, os semigrupos numéricos pertencentes a cadeias infinitas tornam-se raros. Além disso, o número de semigrupos numéricos em cadeias infinitas com multiplicidade de até 7 é explicitamente calculado. Finalmente, a tese estende esses conceitos para o contexto dos semigrupos numéricos generalizados (GNS), que são submonoides de $\mathbb{N}_0^d$ com complemento finito. Um arcabouço para árvores de GNS é explorado e uma caracterização de cadeias infinitas nesse contexto generalizado é apresentada.

List of Symbols

$\#X$ cardinality of set X

$\gcd(X)$ greater commom divisor of X

Λ numerical semigroup

$\langle X \rangle$ generating set of X

$\lceil x \rceil$ least integer greater than or equal to x

$\lfloor x \rfloor$ greatest integer lower than or equal to x

$\mathbb{I}$ set of all infinite chains of numerical semigroups

$\mathcal{L}(\Lambda)$ left elements of Λ

$\mathcal{L}_{\prec}(S)$ left elements of S with respect to $\prec$

$\mathbb{Q}[d]$ polynomial ring in d over rational numbers $\mathbb{Q}$

$\mathbf{0}$ d-uple $(0, \ldots, 0)$

$\mathbf{1}$ d-uple $(1, \ldots, 1)$

$h(\Lambda)$ efficacy of Λ

$H(S)$ set of gaps of S

i_g number of numerical semigroups of genuns g that are in an infinite chain

$i_g(m)$ number of numerical semigroups of genuns g, multiplicity m and which are in an infinite chain

$I_{\mathcal{H}}$ set of all hyperellipitic semigroups

$I_{\mathcal{O}}$ set of all ordinary semigroups

$U_{\prec}(S)$ set of effective generators of S with respect to $\prec$

$X \ni x$ x belongs to X

$X \setminus Y$ difference between sets X and Y

$X \sqcup Y$ disjoint union between sets X and Y

X^C complement of X

$\mathcal{T}^d_{\prec}$ tree of generalized numerical semigroups of dimension d with respect to $\prec$

$\mathbb{S}$ set of all numerical semigroups

$\mathcal{H}_g(\Lambda)$ hyperelliptic semigroup of genus h

$\mathcal{O}(\mathbf{c})$ ordinary generalized numerical semigroup with corner $\mathbf{c}$

$\mathcal{O}_g$ ordinary semigroup with genus g

$\mathbb{N}$ set of positive integers

$\mathbb{N}_0$ set of non-negative integers

$\mathbb{N}_0^d$ d-dimensional Cartesian power of $\mathbb{N}_0$

$\mathbb{N}_{\geqslant 2}$ set of integers greater than or equal to 2

$\mathbf{e}_i$ standard basis in $\mathbb{R}^d$

$\mathbb{R}^d$ d-dimensional Euclidean space

$\mathbb{R}_{\geqslant 0}^d$ d-dimensional Cartesian power of $\mathbb{R}_{\geqslant 0}$

$c(\Lambda)$ conductor of Λ

$e(\Lambda)$ embbeding dimension of Λ

$F(\Lambda)$ Frobenius number of Λ

$F_{\prec}(S)$ Frobenius number of S with respect to $\prec$

F_g g-th Fibonacci number

$g(\Lambda)$ genus of Λ

$h(\Lambda)$ efficacy of Λ

$m(\Lambda)$ multiplicity of Λ

$m_{\prec}(S)$ multiplicity of S with respect to $\prec$

n_g number of numerical semigroups with genus g

S_g set of numerical semigroups with genus g

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Chapter 1

Introduction

Whenever I introduce numerical semigroups, I feel compelled to start with a warning. This mathematical object is deceptively simple yet remarkably captivating. Numerical semigroups have all the qualities of a fascinating mathematical subject. They can be introduced with relatable and entertaining real-life examples, such as McNuggets or ATM withdrawals (see [24]). They have a rich history dating back to the resolution of Diophantine problems in the 19th century (an overview is provided in [2]). They also intersect with other areas of pure and applied mathematics, including commutative algebra (examples can be found in [5]), algebraic geometry (explored in [4]), error-correcting codes (one of the earliest studies appears in [43]), and even music theory (discussed in [14]).

The reason for caution lies in the stark contrast between the simplicity of explaining numerical semigroups and the difficulty of solving problems related to them. In this theory, one can find: problems or conjectures that are simple to understand but that remain unanswered or unproved, some of them for more than 40 years; properties that become tools for different areas of mathematics; and applications to information theory. Throughout this chapter, a survey of

numerical semigroups is provided, focusing on the historical trajectory that leads to the research question of this thesis.

1.1 Context and motivation

A numerical semigroup Λ is a subset of the non-negative integers $\mathbb{N}_0$, containing 0, closed under addition and with finite complement in $\mathbb{N}_0$. As will be seen, basic concepts of numerical semigroups involve terms such as genus, multiplicity, and embedding dimension. The emergence of these terminologies reflects the relationship between numerical semigroups and commutative algebra. The fact that the set of polynomials with exponents in a numerical semigroup S forms a ring, called the semigroup ring of S over a field K, is the starting point for a broad field of research involving semigroups and commutative algebra.

Numerical semigroups also appear in geometry. This connection originates with the Weierstrass Theorem, proved around 1860. Weierstrass showed that, given a smooth, projective algebraic curve X defined over an algebraically closed field K, for every point $P \in X$, one can associate a numerical semigroup formed by the non-negative integers n for which there exists a rational function on X that is regular on $X \setminus P$ and has a pole of order n at P.

An additional application can be found in error-correcting codes, specifically Goppa codes, introduced in the 1980s. These codes are constructed from curves over finite fields, and their parameters can be estimated using results and concepts from algebraic geometry, including the Weierstrass semigroup. The perspective of numerical semigroups has been useful, for example, to improve bounds of parameters such as the length, the correction capability, or the dimension of codes.

The elements of $\mathbb{N}_0$ that are not in Λ are called the *gaps* and the number of gaps is the *genus* $g(\Lambda)$ of the numerical semigroup. The central question from which the specific objectives of this research arise is: Given a nonnegative integer g, what is the number n_g of numerical semigroups with genus g? After computing

the values of n_g for $g \leqslant 50$, Bras-Amorós [11] conjectured the following:

a. $n_g \geqslant n_{g-1} + n_{g-2}$, for $g \geqslant 2$;

b. $\lim_{g\to\infty} \frac{n_{g-1}+n_{g-2}}{n_g} = 1$;

c. $\lim_{g\to\infty} \frac{n_g}{n_{g-1}} = \varphi$, where φ is the golden ratio.

This conjecture implies that the sequence counting numerical semigroups by genus behaves similarly to the Fibonacci sequence. Since then, significant progress has been made in understanding the growth of the sequence n_g.

In terms of computational advances, besides the pioneering work of Bras-Amorós in 2008 [11], other significant contributions have been made to the computation of this sequence. In 2016, Fromentin and Hivert [37] extended the computations to n_{70} using the tree of numerical semigroups. Later, Bras-Amorós and Fernández-González [17] determined n_{71} and n_{72}, followed by Delgado, Eliahou, and Fromentin [31], who established the values of n_{73}, n_{74}, and n_{75} in 2025. More recently, in preprint [21], Bras-Amorós reported the terms n_{76} and $n_{77} = 47\ 008\ 818\ 196\ 495\ 180$, the latest known value in the sequence to date. This sequence is also documented in the Online Encyclopedia of Integer Sequences [54]. Additionally, Delgado, García-Sánchez, and Morais have developed an algorithm to enumerate all numerical semigroups of a given genus, implemented in the *NumericalSgps* package for the computer algebra system GAP [38, 32].

In 2007, Bras-Amorós and de Mier [16] used Dyck paths to prove $n_g \leqslant C_g$, where C_g is the Catalan number of order g. Bras-Amorós [12], in 2009, provided new upper and a lower bounds for n_g, specifically $2F_g \leqslant n_g \leqslant 1 + 3 \cdot 2^{g-3}$, where F_g denotes the g-th Fibonacci number. She used some techniques in multisets and the semigroup tree. These bounds were further refined in 2010 by Elizalde [34] using generating functions.

Zhao [57] explored the Fibonacci-like properties proposed by Bras-Amorós and, in 2010, published a paper showing that, among other results, most numerical semigroups with a fixed genus have a Frobenius number less than three times their multiplicity. This result served as the primary tool for Zhai [56], who, in 2013, published the proofs of Bras-Amorós's Conjectures b and c.

Thus, of the three conjectures proposed by Bras-Amorós regarding the behavior of the sequence n_g, only Conjecture a remains unresolved. In fact, even a weaker version of the conjecture, namely $n_{g-1} \leqslant n_g$, is still an open problem. The result of [56] ensures that, starting from a certain genus, Conjecture a holds. Additionally, computational results verify the conjecture for values of g up to 76.

As can be observed in [12], one main tool for investigating the sequence n_g is that of the tree of numerical semigroups. This tree was addressed in [50, 51, 48, 12]. The root of this tree is the trivial numerical semigroup of genus 0 ($\mathbb{N}_0$), and at each level of the tree there are the semigroups whose genus equals the level. The parent of each non-trivial semigroup in the tree is the semigroup obtained by adding to the semigroup its largest gap, the so-called *Frobenius number* of the semigroup. This way, if one can explore the whole tree of semigroups, computing the number of semigroups of a given genus is equivalent to counting the number of nodes of the tree at the given depth. For more details on this tree, the reader can refer to [56, 37].

Several authors have sought to derive formulas for the number of semigroups of genus g with an additional fixed parameter, such as:

- the Frobenius number [10, 9];
- the *multiplicity*, i.e., the smallest nonzero element of the semigroup [9, 41, 39, 3];
- the *efficacy*, defined as the number of children in the tree [45];

- the number of even gaps [8];
- the *ordinarization number* [13], which measures the distance, in the ordinarization tree, from the semigroup to its ordinary ancestor with the same multiplicity. An *ordinary semigroup* is one whose set of gaps consists of consecutive integers in $\mathbb{N}$;
- the *quasi-ordinarization number* [23], which measures the distance, in the quasi-ordinarization tree, from the semigroup to its quasi-ordinary ancestor with the same multiplicity. A *quasi-ordinary semigroup* is defined as a semigroup that has exactly one gap greater than its multiplicity.

The work of Failla, Peterson, and Utano in [35] introduces a direct generalization of numerical semigroups in $\mathbb{N}_0$ to submonoids of $\mathbb{N}_0^d$. This object had already been explored before, but within other contexts that did not directly associate it as a generalization of numerical semigroups [46]. A monoid $S \subseteq \mathbb{N}_0^d$ is said to be a generalized numerical semigroup (GNS) if its set of gaps, $H(S) = \mathbb{N}_0^d \setminus S$, is finite. As for numerical semigroups, the number of gaps of a GNS is its genus.

Much effort has been put into extending the definitions of the parameters of numerical semigroups to generalized ones. The main challenge for this is the fact that many of these definitions depend on the total order relation between the elements of the semigroup, an order that is natural when the setting is $\mathbb{N}_0$. However, in $\mathbb{N}_0^d$ the natural order relation is partial.

In the paper [35] the authors use a relaxed monomial order to define the multiplicity, the Frobenius number and the effective generators of a GNS. In [27], the concept of ordinary GNS is introduced. In [7], Bernardini, Tenório, and Tizziotti generalized the concept of conductor of a numerical semigroup to the *corner* of a generalized numerical semigroup, using the natural partial order relation inherited from $\mathbb{N}$. They also provide results on the relation between genus and corner that extend similar ones from the numerical semigroup theory. The conditions under

which a set generates a semigroup are extended for GNS in [28]. This characterization depends on the behavior of the elements of a set $A \subseteq \mathbb{N}_0^d$ on the d axes and on the d half-lines parallel to the axes one unit apart.

Still in [35], the GNS tree, or rather, the GNS trees, were constructed. This is because, as in the case of numerical semigroups, a tree that organizes all GNSs by genus depends on the concept of effective generators which, in turn, depends on the total order relation. That is, for each relaxed monomial order $\prec$ defined in $\mathbb{N}_0^d$, there is a tree $\mathcal{T}_{\prec}^d$ of GNSs. It is important to emphasize that, despite the existence of several GNS trees for each fixed dimension, the number of nodes at each level g of the tree does not vary between one tree and another. In this sense, for any relaxed monomial order $\prec$ in $\mathbb{N}_0^d$, the number of nodes at level g of the tree $\mathcal{T}_{\prec}^d$ is equal to the number $n_{g,d}$ of GNSs in dimension d with genus g.

It was following the spirit of Bras-Amorós conjectures that they introduced the investigation of the asymptotic behavior of sequence $n_{g,d}$. In 2016, Failla, Peterson and Utano [35] compute $n_{g,d}$ for $g = 1, 2, 3$, namely,

$$n_{1,d} = d, \qquad n_{2,d} = \frac{3}{2}d^2 + \frac{1}{2}d, \qquad n_{3,d} = \frac{5}{3}d^3 + \frac{5}{2}d^2 - \frac{1}{6}d.$$

According to [35], $n_{g,d}$ is a polynomial function of d with degree g. In 2021, Cisto, Delgado, and García-Sánchez [26] displayed the polynomials $n_{g,d}$ in d of g ranging from 4 to 9. They also provided values of $n_{g,d}$ for d up to 10 and g in some cases up to 21. In the preprint [44] from 2022, Li gave a lower bound for $n_{g,d}$.

In the numerical semigroup tree, there exist nodes with infinite descendants, such as ordinary semigroups. An *infinite chain* in the semigroup tree is an infinite sequence of subsequently included numerical semigroups that are adjacent in the semigroup tree. In contrast, leaves represent endpoints in the tree structure, and this is precisely what makes its growth unpredictable. Infinite chains were first analyzed in [15], where Bras-Amorós and Bulygin characterized such nodes using the greatest common divisor of the first elements of the numerical semigroup.

1.2 Objectives

Computational results up to the first eleven levels of the tree indicate that infinite chains are rare; however, without them, the tree would not be infinite. For this reason, this research focuses on infinite chains as a means to investigate the broader problem of the tree growth. At some point, the multiplicity is fixed in order to go deeper into the investigation. Some of these findings are also to extend on infinite chains of numerical semigroups to the GNS framework.

1.3 Thesis structure

This work is composed of seven chapters.

It begins with the current **Chapter 1**, which presents a discussion of the state of the art in the research area and a description of the research problem. The remaining chapters are organized as follows.

In **Chapter 2** some preliminary results on numerical semigroups are stated, which constitute the root of this work. This chapter also introduces the tree framework for numerical semigroups.

In **Chapter 3**, the main result from [15] is revisited, and an imprecision regarding the cardinality of the set of infinite chains to which a semigroup belongs is corrected in the case where the greatest common divisor of the left elements is a prime number. The fixed result is presented in Theorem 3.5. Furthermore, it is proven that for each genus $g \geqslant 5$, there are more semigroups of that genus that do not belong to infinite chains than those that do (Theorem 3.15). Moreover, infinite chains in subtrees with fixed multiplicity are examined. When the multiplicity is a prime number, it can be shown, as a consequence of Theorem 3.5, that there is only one infinite chain in the tree of semigroups with such multiplicity. For the cases of multiplicity 4 and 6, a self-replication behavior is identified in the subtree formed by semigroups with fixed multiplicity 4 that belong to infinite

chains (Subsection 3.2.2), while a different self-replication behavior is observed in the subtree formed by semigroups with fixed multiplicity 6 that belong to infinite chains (Subsection 3.2.3). These findings allow for the derivation of a formula for the number of semigroups of genus g that belong to infinite chains, for any given prime multiplicity (Theorem 3.18), as well as for multiplicities 4 (Theorem 3.19) and 6 (Theorem 3.27).

Recent references analyze the expected multiplicity in terms of the genus of a semigroup as the genus grows to infinity, as well as the expected Frobenius number in terms of the multiplicity [42, 53]. Indeed, let $\gamma = (5 + \sqrt{5})/10$. Fix $\varepsilon > 0$. It is stated in [42, Proposition 16] that the portion of semigroups of genus g with multiplicity between $(\gamma - \varepsilon)g$ and $(\gamma + \varepsilon)g$ with respect to the whole set of numerical semigroups of genus g approaches 1 as the genus g grows to infinity. On the other hand, [42, Theorem 4(1)] tells that the portion of semigroups of genus g with Frobenius number F satisfying $(2 - \varepsilon)m < F < (2 + \varepsilon)m$, where m is the multiplicity of the semigroup, with respect to the whole set of numerical semigroups of genus g approaches 1 as the genus g grows to infinity. See also [53, Theorem 8]. These results are used in **Chapter 4** to demonstrate that, as the genus increases, semigroups in infinite chains become rare in the tree.

Chapter 5 is devoted to arranging the roots of generalized numerical semigroups. Several examples of relaxed monomial orders are presented, and a tree is illustrated.

A characterization of GNSs belonging to infinite chains is provided (Theorem 6.1) in **Chapter 6**. Several examples are presented, some of which (Figure 6.3) demonstrate that a GNS can belong to an infinite chain in one tree $\mathcal{T}^d_{\prec_1}$ while being part of a finite chain in another tree $\mathcal{T}^d_{\prec_2}$.

Finally, in **Chapter 7**, a summary of the main findings of this work is provided. Additionally, the connections between the results obtained and previous research are discussed. The quasi-polynomial nature of the counting function for

semigroups in infinite chains suggests further connections with other approaches to semigroup counting. Open problems and potential directions for future research are also pointed out.

1.4 Contributions

This thesis gathers all the results obtained during the development of my Ph.D research project. Some of these results have been published in the last year of this project, but not all. Chapter 3 follows the published paper

- [52] M. Rosas-Ribeiro and M. Bras-Amorós. Infinite chains in the tree of numerical semigroups. In *Semigroup Forum*, pages 1–20. Springer, 2024.

Chapter 4 comes from the submitted paper

- [18] M. Bras-Amorós and M. Rosas Ribeiro. Rarity of the infinite chains in the tree of numerical semigroups. *arXiv preprint arXiv:2401.18060*, 2024.

Chapter 6 is a paper in progress.

Chapter 2

Numerical Semigroups: roots and frameworks

> Mandei plantar folhas de sonhos no jardim do solar, as folhas sabem procurar pelo sol e as raízes procurar, procurar.
>
> Os Mutantes, Brazilian band, in the song Panis Et Circenses

The foundation of this research lies in numerical semigroups, a subject likely originating from the study of the set of non-negative solutions of the Diophantine equations [2]. Explicitly, given positive integers $a_1, \ldots, a_n$ with the greatest common divisor equal to one, the set of elements $b \in \mathbb{N}$ for which the equation $a_1x_1 + \cdots + a_nx_n = b$ has a non-negative integer solution forms a numerical semigroup. Arguably the most famous problem that inaugurated an area of investigation in numerical semigroups is the Frobenius problem: what is the largest integer b for which there is no non-negative integer solution to the former equation?

Examples of this problem have been presented for the purpose of mathematical outreach, with some well-known cases involving withdrawing amounts from ATMs and fulfilling orders for boxes of McNuggets at McDonald's franchises [24].

In this chapter, we present fundamental concepts of numerical semigroups and key structures that will support our study. Many of these concepts can be found in [49]. We also introduce the numerical semigroup tree, a central tool for understanding the growth of the sequence n_g.

2.1 The roots for numerical semigroups

In this section, we will present concepts and results on numerical semigroups that are generally well-known and will serve as the basis for the subsequent chapters. We will also underline, through examples, certain classes of semigroups that will be equally important throughout this work.

Definition 2.1. A *numerical semigroup* Λ is a cofinite submonoid of $\mathbb{N}_0$.

That is to say, a numerical semigroup Λ is a subset of the non-negative integers $\mathbb{N}_0$, containing 0, closed under addition and with finite complement in $\mathbb{N}_0$.

Definition 2.2. The elements of $\mathbb{N}_0$ that are not in a numerical semigroup Λ are called the *gaps* and the number of gaps is the *genus* $g(\Lambda)$ of the numerical semigroup. We will also denote by $H(\Lambda)$ the set of gaps of Λ.

Mathematicians have long pondered the question: given a non-negative integer g, how many numerical semigroups have genus g? Let S_g denote the set of numerical semigroups of genus g. By the next proposition, the set S_g is finite.

Proposition 2.3. *[6, Corollary 1.1] If $g \geqslant 1$, then $\#S_g \leqslant \binom{2g-2}{g-1}$.*

In 2007, Bras-Amorós [11] conjectured that the sequence n_g, which counts the number of elements of S_g, grows like the Fibonacci sequence. Of the three

conjectures proposed at the time, only one remains unresolved: the assertion that $n_{g-2}+n_{g-1} \leqslant n_g$. Even a weaker version, $n_{g-1} \leqslant n_g$, has yet to be proven. As we will see in Section 2.2, n_g clearly exhibits (significant) growth, but no formal proof has been established so far. This is the core problem that inspires this research.

Example 2.4. The set $\mathbb{N}_0$ is the unique numerical semigroup with no gaps, hence $n_0 = 1$. To create a gap in $\mathbb{N}_0$, there is only one possibility: transforming the element 1 into a gap. Removing (only) any other element a would result in the issue that a can be expressed as $\underbrace{1+1+\cdots+1}_{a \text{ times}}$, and the set would cease to be a monoid. Thus, $\mathbb{N}_0 \setminus \{1\}$ is the only numerical semigroup with one gap, and therefore $n_1 = 1$. Similarly, we find that $\mathbb{N}_0 \setminus \{1,2\}$ and $\mathbb{N}_0 \setminus \{1,3\}$ are the only semigroups of genus two, that is, $n_2 = 2$.

Evidently, this type of reasoning is insufficient to compute n_g for all $g \geqslant 0$. Some additional terms of the sequence n_g are: 1, 1, 2, 4, 7, 12, 23, 39, 67, 118, 204, 343, 592, 1001, 1693, 2857, 4806, 8045, 13467, 22464, 37396, 62194, 103246 [54].

One way to label all elements of the numerical semigroup Λ is to enumerate them.

Definition 2.5. The *enumeration* λ of a numerical semigroup Λ is the unique increasing bijection between $\mathbb{N}_0$ and Λ, and we denote by λ_i the image of $i \in \mathbb{N}_0$. In particular, $\lambda_0 = 0$. The *multiplicity* of Λ is $m(\Lambda) = \lambda_1$, that is, the smallest nonzero element of Λ.

Let $g := g(\Lambda)$, and suppose that $m(\Lambda) = \lambda_1$ is greater than $g(\Lambda)+1$. In this case, there would be more than g gaps between λ_0 and λ_1, which contradicts the definition of g. This observation leads us to the next lemma.

Lemma 2.6. *For a numerical semigroup Λ, it holds $m(\Lambda) \leqslant g(\Lambda)+1$.*

Next we define some more basic elements of a numerical semigroup. The concept of conductor is inherited from algebraic geometry, the concept of Frobenius number comes from the historical problem that Frobenius addressed and the concept of left elements was introduced by Eliahou in [33].

Definition 2.7. The *conductor* $c(\Lambda)$ of a numerical semigroup Λ is defined as $c(\Lambda) = \lambda_k$, where $\lambda_k + n \in \Lambda$ for all $n \in \mathbb{N}$ and $\lambda_k - 1 \notin \Lambda$. The number $c(\Lambda) - 1$ is the largest gap and is called the *Frobenius number* of Λ and denoted by $F(\Lambda)$. The elements of Λ that are smaller than its conductor are referred to as the *left elements* (*or small elements*), denoted by $\mathcal{L}(\Lambda)$.

It is easy to check that if c and g are the conductor and the genus of Λ, respectively, then $\lambda_{c-g} = c$ and for $\lambda_i \geqslant c$, $\lambda_i = i + g$.

Example 2.8. Let $[a, \infty)$ stand for $\{i \in \mathbb{N}_0 : i \geqslant a\}$. Numerical semigroups of the form $\{0\} \cup [\lambda_1, \infty)$ are called *ordinary*, a term introduced by Bras-Amorós in [19]. This nomenclature is inspired by the fact that, in algebraic geometry, almost all rational points on a curve of genus g over an algebraically closed field have a Weierstrass semigroup of the form $\{0\} \cup [g + 1, \infty)$, and such points are referred to as ordinary [36, 40, 55]. Similarly, semigroups of the form $\{2n : n \geqslant 0\} \cup [\lambda_k, \infty)$, λ_k even, are called *hyperelliptic semigroups*.

Note that the left elements and the conductor of a numerical semigroup together uniquely determine the semigroup. Also, it is worth noting that the set $\mathcal{L}(\Lambda)$ will play a significant role in the study of infinite chains of numerical semigroups.

Definition 2.9. A subset $A = \{a_1 < a_2 < \cdots < a_n\}$ of a semigroup Λ is a *generating set* of Λ if every element of Λ can be expressed as a sum of elements in A. In this case, we write $\Lambda = \langle a_1, \ldots, a_n \rangle$. A generating set of a numerical semigroup is called *minimal* if no proper subset of it generates the semigroup.

Theorem 2.10. *[49, Theorema 2.7] Every numerical semigroup admits a unique minimal set of generators. This minimal set of generators is finite.*

The next lemma characterizes when a set generates a numerical semigroup.

Lemma 2.11. *[49, Lemma 2.1] Let A be a nonempty subset of $\mathbb{N}$. Then the set $\langle A\rangle$ generated by A is a numerical semigroup if and only if $gcd(A) = 1$.*

Definition 2.12. The cardinality of the minimal system of generators of a numerical semigroup Λ is called the *embedding dimension* of Λ and we denote it by $e(\Lambda)$.

In view of the fact that two minimal generators cannot lie in the same congruence class modulo $m(\Lambda)$, one has the next proposition.

Proposition 2.13. *[49, Proposition 2.10] Let Λ be a numerical semigroup and λ its enumeration, then $e(\Lambda) \leqslant \lambda_1$.*

The next definition was set by Bras-Amorós and Bulygin in [15].

Definition 2.14. The minimal generators of a numerical semigroup Λ that are not left elements are the *effective generators* (or *right generators*) of Λ.

Example 2.15. Let $\Lambda = \langle 4, 9, 10, 11\rangle$. We will often use the representation shown in Figure 2.1 to represent numerical semigroups. All the gaps are shaded in gray, while the effective generators are highlighted in orange. About Λ we underscore:

Figure 2.1: Representation of $\Lambda = \langle 4, 9, 10, 11\rangle$.



$g(\Lambda) = 6$, $m(\Lambda) = 4$, $c(\Lambda) = 8$, $F(\Lambda) = 7$, $\mathcal{L}(\Lambda) = \{0, 4\}$ and $e(\Lambda) = 4$.

Proposition 2.16. *[49, Proposition 7.28] Let Λ be a numerical semigroup, and $x \in \Lambda$. Then $\Lambda \setminus \{x\}$ is a numerical semigroup if and only if x is a generator in Λ.*

Proposition 2.17. *[49, Proposition 4.33] Let Λ be a numerical semigroup. Then $\Lambda \cup \{F(\Lambda)\}$ is a numerical semigroup; moreover, $F(\Lambda)$ is an effective generator in $\Lambda \cup \{F(\Lambda)\}$.*

By manipulating the effective generators, we will construct the numerical semigroup tree, which is the focus of the next section. The following proposition establishes an upper bound for the maximum effective generator.

Proposition 2.18. *Every effective generator of Λ is at most $c(\Lambda) + m(\Lambda) - 1$.*

Proof. Indeed, for a numerical semigroup Λ, if $a \geqslant c(\Lambda)+m(\Lambda)$, then $a = m(\Lambda)+(a - m(\Lambda))$, with both $m(\Lambda)$ and $a - m(\Lambda)$ non-gaps. □

As discussed in the Introduction (Chapter 1), some authors have obtained results regarding the asymptotic behavior of semigroups with specific characteristics. For instance, Zhao [57] demonstrated that semigroups with Frobenius number smaller than three times their multiplicity are predominant as the genus increases. Similarly, Kaplan and Ye [42] reached the same conclusion for semigroups whose multiplicity is restricted to a given interval. To wrap up these preliminaries, we state two lemmas that will serve as tools for Chapter 4.

Let $\gamma = (5 + \sqrt{5})/10$, let $\varepsilon_1, \varepsilon_2$ be real numbers, and let Λ be a numerical semigroup. Define

$$\begin{aligned} \mathcal{M}_g(\varepsilon_1) &= \{\Lambda \in S_g : (\gamma - \varepsilon_1)g < m(\Lambda) < (\gamma + \varepsilon_1)g\}, \\ \mathcal{F}_g(\varepsilon_2) &= \{\Lambda \in S_g : (2 - \varepsilon_2)m(\Lambda) < F(\Lambda) < (2 + \varepsilon_2)m(\Lambda)\}. \end{aligned}$$

Lemma 2.19. *[42, Proposition 16] For any fixed $\varepsilon_1 > 0$, $\lim_{g\to\infty} \frac{\#\mathcal{M}_g(\varepsilon_1)}{n_g} = 1$.*

Lemma 2.20. *[42, Theorem 4(1)] For any fixed $\varepsilon_2 > 0$, $\lim_{g\to\infty} \frac{\#\mathcal{F}_g(\varepsilon_2)}{n_g} = 1$.*

2.2 The tree of numerical semigroups

One approach to studying the growth of the sequence n_g is through the numerical semigroup tree [50, 51, 48, 12]. This tree systematically organizes all numerical semigroups, with $\mathbb{N}_0$ (the semigroup of genus 0) as its root. Each level g of the tree corresponds to the numerical semigroups of genus g. In this section, we detail the construction of this tree, examine key components, and underline infinite branches that emerge from it.

We will intuitively develop the construction of the numerical semigroup tree. This process relies on defining a function that maps the set S_g to S_{g-1}. Since for any $\Lambda \in S_g$, the set $\Lambda \cup \{F(\Lambda)\} \in S_{g-1}$, Proposition 2.17 provides a direct way to define this mapping. However, to systematically construct the tree starting from the root $\mathbb{N}_0$, a procedure is needed to apply to S_{g-1} and generate S_g. Proposition 2.16 suggests removing the generators of each semigroup in S_{g-1}. Nevertheless, as illustrated in Example 2.21, doing so introduces redundancies, meaning that the number of elements at level g of this structure may not correspond to n_g. Considering the function $f : S_g \to S_{g-1}$, defined by $f(\Lambda) = \Lambda \cup \{F(\Lambda)\}$, if we systematically remove the generators greater than $F(\Lambda)$ from each $\Lambda \in S_g$, that is, the effective generators, we obtain a tree that organizes all numerical semigroups, with semigroups of genus g occupying each level g of the tree.

Example 2.21.

$S_6 \ni \langle 4, 5\rangle$:



$S_6 \ni \langle 4, 9, 10, 11\rangle$:



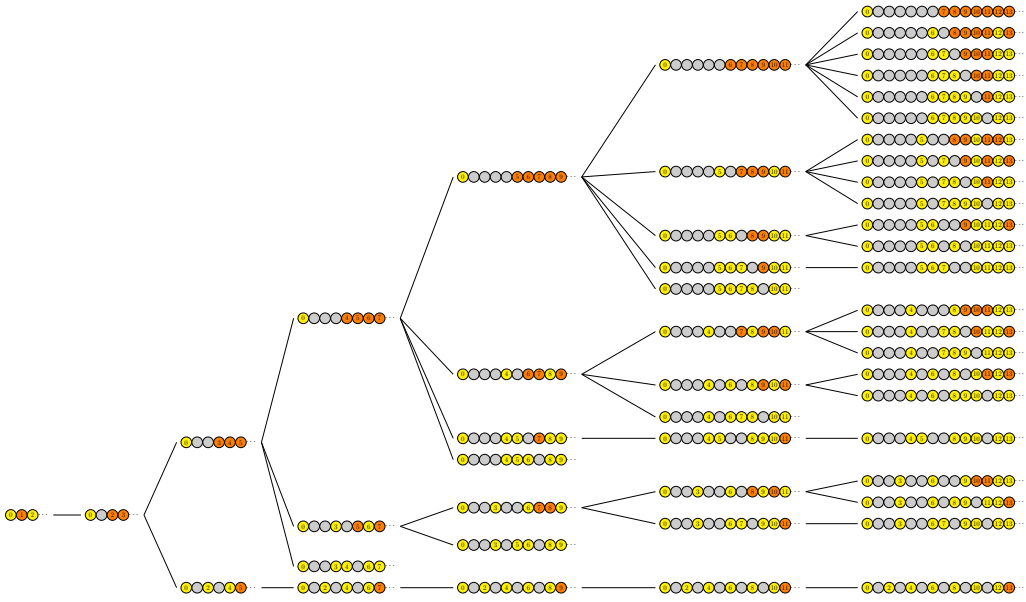
$S_7 \ni \langle 4, 5\rangle \setminus \{5\} = \langle 4, 9, 10, 11\rangle \setminus \{11\}$:



In Figure 2.2 one can see the tree with numerical semigroups of level at most 6. In this representation the semigroups were lexicographicaly ordered with respect to the gaps. This means that at each level g of the tree, the set S_g is arranged from

top to bottom with the usual order of $\mathbb{N}$ applied to the gaps of each semigroup. For more details of this tree, we suggest consulting [15, 56, 37] and for more details about tree ordering, [17, 22]. From the tree perspective, numerical semigroups can also be seen as nodes. In this way, the *children* of a numerical semigroup Λ are those numerical semigroups that are obtained by taking away one right generator from Λ. Conversely, the *parent* of a numerical semigroup is obtained by adding its Frobenius number (the parent is one level lower in the tree).

Figure 2.2: Tree of numerical semigroups with the nodes of level at most 6.



Definition 2.22. The number of children of a node Λ is called its *efficacy* and is denoted by $h(\Lambda)$. Every numerical semigroup of genus larger than $g(\Lambda)$ connected to Λ through edges in the tree is said to be a *descendant* of Λ. A node that has no descendants is said to be a *leaf* and a node that has exactly one child is said to be a *stick*.

By Lemma 2.6 and Proposition 2.13, one can deduce that $h(\Lambda) \leqslant e(\Lambda) \leqslant m(\Lambda) \leqslant g(\Lambda) + 1$.

The next lemma was proved in [12] but we enunciate it here in a different way, underlying the result used in the original proof. Among other things, it allows us to study the children of the children, or *grandchildren*, of most numerical semigroups.

Lemma 2.23. *[12, Lemma 3] If $\lambda_{i_1} < \lambda_{i_2} < \cdots < \lambda_{i_n}$ are the effective generators of a non-ordinary numerical semigroup Λ, then the effective generators of $\Lambda \setminus \{\lambda_{i_j}\}$ are either $\lambda_{i_{j+1}}, \ldots, \lambda_{i_n}$ or $\lambda_{i_{j+1}}, \ldots, \lambda_{i_n}, \lambda_{i_j} + \lambda_1$.*

As we will see in the main results of Chapter 3, ordinary semigroups are not covered by these results. This stems from the distinct behavior of their effective generators compared to other semigroups. The next lemma conveys this idea.

Lemma 2.24. *[12, Lemma 4] The ordinary semigroup $\Lambda = \{0, g+1, g+2, \ldots\}$ has minimal set of generators $\{g+1, g+2, \ldots, 2g+1\}$ and*

a. $\Lambda \setminus \{g+1\}$ has $g+2$ effective generators;

b. $\Lambda \setminus \{g+2\}$ has g effective generators;

c. $\Lambda \setminus \{g+r\}$, with $r > 2$, has $g - r + 1$ effective generators.

Since $g + 1 > 1$ for any genus g, the collection $\{\mathbb{N}_0 \setminus \{1, \ldots, g+1\}\}_{g \geqslant -1}$ constitutes an infinite set of ordinary semigroups, where $\mathbb{N}_0 \setminus \{1, \ldots, g+1\}$ is the parent of $\mathbb{N}_0 \setminus \{1, \ldots, g+2\}$. As illustrated in Figure 2.2, this collection forms an infinite chain of semigroups distinctly located. Using the lexicographic order in the gaps, this chain is in the upper region of the tree and if we use the lexicographic order in the non-gaps, this chain will appear in the lower part of the tree. On the other hand, hyperelliptic semigroups, generated by $\{2, 2g+1\}$, have, for $g \geqslant 2$, only one effective generator, which is the non-gap $2g + 1$. Thus, they

are sticks and also form an infinite chain, this time in the lower part of the tree when we use lexicographic order in gaps and in the upper part of the tree when we use lexicographic order in non-gaps.

Another aspect that is evident in the tree is that semigroups of the same multiplicity form subtrees. A tool for working with numerical semigroup problems is to fix the multiplicity, as Kaplan did in [41]. Specifically, if each subtree with a fixed multiplicity exhibits an increasing behavior, then the entire tree will also exhibit this behavior. In Section 3.2 of Chapter 3, we will examine some of these subtrees in order to study the infinite chains within them.

The last preliminary result is a characterization of ordinary semigroups by the number of children in terms of its genus.

Lemma 2.25. *If a numerical semigroup Λ with genus g has $g+1$ children, then Λ is ordinary.*

Proof. The following sequence of inequalities must indeed be a sequence of equalities:

$$g + 1 = h(\Lambda) \leqslant e(\Lambda) \leqslant m(\Lambda) \leqslant g + 1.$$

Hence, $m(\Lambda) = g + 1$, meaning that all gaps of Λ are smaller than $m(\Lambda)$, and therefore Λ must be ordinary. □

Chapter 3

Infinite Chains in the Tree of Numerical Semigroups

> Frente ao infindo, costas contra o
> planeta, já sou a seta sem direção.
>
> Caetano Veloso, Brazilian singer
> and musician, in the song
> Madre Deus

Within this chapter we are interested in studying the nodes of the tree of numerical semigroups that have infinitely many descendants. Infinite chains, first studied in [15], are defined as a sequence of semigroups $\Lambda_0, \Lambda_1, \Lambda_2, \ldots$ such that $\Lambda_0 = \mathbb{N}_0$ and Λ_i is the parent of Λ_{i+1} in the semigroup tree. Notice that a numerical semigroup belongs to an infinite chain if and only if it has infinitely many descendants.

First, we analyze which nodes in the semigroup tree have an infinite number of descendants. For the nodes having a finite number of descendants we give a way

to determine the descendant at largest distance; for the nodes having an infinite number of descendants we determine the number of infinite chains in which the semigroup lies. It turns out here that primality and coprimality of integers appear in the scene as discriminating factors.

Let $\mathbb{S}$ be the set of all numerical semigroups, and let $\mathbb{I}$ be the set of all infinite chains. One element of $\mathbb{I}$ contains infinitely many elements in $\mathbb{S}$. As an example, the sequence of ordinary semigroups, $\mathcal{O}_g = \{0, g+1, \to\}$ for any $g \in \mathbb{N}_0$ (the symbol $\to$ means that every integer greater than $g+1$ belongs to the set), is in $\mathbb{I}$. We denote it $I_{\mathcal{O}}$. That is,

$$I_{\mathcal{O}} = \{\mathcal{O}_g : g \geqslant 0\}.$$

Another example is the sequence of hyperelliptic semigroups, $\mathcal{H}_g = \{0, 2, 4, \ldots, 2g, \to\}$. We denote it $I_{\mathcal{H}}$. That is,

$$I_{\mathcal{H}} = \{\mathcal{H}_g : g \geqslant 0\}.$$

Each element in $\mathbb{S}$ may be contained in none, one, several, or infinitely many elements of $\mathbb{I}$. The aim of this section is analyzing what circumstances give what of these cases. Then, we analyze what kind of generators appear in infinite chains.

For the proof of the next lemma we use that the integers $\ell_1, \ldots, \ell_m$ generate a numerical semigroup if and only if they are coprime (Lemma 2.11).

Lemma 3.1. *Given an infinite chain $I = (\Lambda_i)_{i\geqslant 0}$ different than $I_{\mathcal{O}}$, it holds that*

$$\bigcap_{i\geqslant 0} \Lambda_i = d \cdot \Lambda$$

for some integer $d > 1$ and some numerical semigroup Λ.

Proof. The intersection $\bigcap_{i\geqslant 0} \Lambda_i$ satisfies $0 \in \bigcap_{i\geqslant 0} \Lambda_i$. Since there exists a non-ordinary semigroup in I, there must be a non-zero element $x \in \bigcap_{i\geqslant 0} \Lambda_i$ and

$x + y \in \bigcap_{i \geqslant 0} \Lambda_i$ for all $x, y \in \bigcap_{i \geqslant 0} \Lambda_i$.

Furthermore, all elements in $\bigcap_{i \geqslant 0} \Lambda_i$ must be divisible by an integer $d > 1$. Indeed, otherwise we could find a finite set of coprime elements which would generate a numerical semigroup, and this numerical semigroup should be a subset of $\bigcap_{i \geqslant 0} \Lambda_i$. Then the infinite chain would not contain any semigroup with genus larger than that of this semigroup, giving a contradiction. Let d be the greatest of the common divisors of $\bigcap_{i \geqslant 0} \Lambda_i$. Then $\frac{1}{d} \left(\bigcap_{i \geqslant 0} \Lambda_i \right)$ must be a numerical semigroup. □

Lemma 3.2. *Given an integer $d > 1$ and a numerical semigroup Λ, the infinite chain obtained by deleting repetitions in the sequence $\Lambda_j = d \cdot \Lambda \cup \{\ell \in \mathbb{N} : \ell \geqslant j\}$ has intersection $d \cdot \Lambda$.*

Lemma 3.1 and its proof suggest the map

$$\begin{array}{rcl} \sigma : \mathbb{I} \setminus I_{\mathcal{O}} & \longrightarrow & \mathbb{N}_{\geqslant 2} \times \mathbb{S} \\ \{\Lambda_i\}_{i \in \mathbb{N}_0} & \mapsto & (\gcd(\bigcap_{i \geqslant 0} \Lambda_i)), \bigcap_{i \geqslant 0} \Lambda_i / \gcd(\bigcap_{i \geqslant 0} \Lambda_i)). \end{array}$$

Lemma 3.2 proves that the map

$$\begin{array}{rcl} \omega : \mathbb{N}_{\geqslant 2} \times \mathbb{S} & \longrightarrow & \mathbb{I} \setminus I_{\mathcal{O}} \\ (d, \Lambda) & \mapsto & \{d \cdot \Lambda \cup \{\ell \in \mathbb{N} : \ell \geqslant i\}\}_{i \notin d \cdot \Lambda} \end{array}$$

is the inverse of σ.

Consequently, $\mathbb{I} \setminus I_{\mathcal{O}}$ and $\mathbb{N}_{\geqslant 2} \times \mathbb{S}$ are in a one-to-one correspondence. For example, in this correspondence, the image of $I_{\mathcal{H}}$ would be $(2, \mathbb{N}_0)$.

Remark 3.3. Any ordinary numerical semigroup belongs to infinitely many infinite chains. In fact, $\{0, m, \rightarrow\}$ belongs, among others, to $\omega(m', \mathbb{N}_0)$ for any $m' \geqslant m$.

In the next theorem we show that the greatest common divisor of the left elements of a non-ordinary numerical semigroup determines whether the numerical semigroup has an infinite number of descendants.

Definition 3.4. We say that a descendant of a numerical semigroup is a descendant *beyond* a given element a of its parent if it contains all semigroup elements of the parent up to a.

For example, $\{0, 4, 5, 8, 9, 10, 12, \rightarrow\}$ and $\{0, 4, 5, 6, 8, \rightarrow\}$ are descendants of $\{0, 4, \rightarrow\}$, but only $\{0, 4, 5, 6, 8, \rightarrow\}$ is a descendant beyond 6. Also we recall that the integer part of a real number x, denoted by $\lfloor x \rfloor$, is the greatest integer which is lower than or equal to x. Conversely, the ceiling of x, denoted by $\lceil x \rceil$, is the least integer greater than or equal to x.

Theorem 3.5. *Let Λ be a non-ordinary numerical semigroup with enumeration λ, genus g, and conductor c, and let d be the greatest common divisor of $\mathcal{L}(\Lambda)$. Then,*

a. *Λ lies in an infinite chain if and only if $d \neq 1$.*

b. *If $d = 1$, then the descendant of Λ with largest genus is the numerical semigroup generated by the non-zero elements of $\mathcal{L}(\Lambda)$, that is, the numerical semigroup generated by $\lambda_1, \ldots, \lambda_{c-g-1}$.*

c. *If $d \neq 1$ and d is not prime, then Λ lies in infinitely many infinite chains.*

d. *If d is a prime, then the number of infinite chains in which Λ lies is one plus the number of descendants of $\Lambda/d := \{0, \frac{\lambda_1}{d}, \ldots, \frac{\lambda_{c-g-1}}{d}\} \cup \{\ell \in \mathbb{N}_0 : \ell \geqslant \lceil \frac{c}{d} \rceil\}$ beyond $\frac{\lambda_{c-g-1}}{d}$.*

Proof. a. If $c-g-1 = 1$, then Λ belongs at least to the infinite chain $\omega(\lambda_1, \mathbb{N}_0)$, while $d = \lambda_1 \neq 1$. So, we can assume that $c - g - 1 \geqslant 2$.

If $d = 1$ and $c-g-1 \geqslant 2$, then $\lambda_1, \ldots, \lambda_{c-g-1}$ generate a numerical semigroup Λ' with no descendants in the semigroup tree. Now, each descendant of Λ must contain Λ'. Thus, the maximum of the genera of the descendants of Λ is the genus of Λ', which is finite, and so there is no infinite chain containing Λ.

On the other hand, if $d \neq 1$ and $c - g - 1 \geqslant 2$, then

$$\lambda_0 = d\tilde{\lambda}_0, \lambda_1 = d\tilde{\lambda}_1, \ldots, \lambda_{c-g-1} = d\tilde{\lambda}_{c-g-1}$$

with $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{c-g-1}$ coprime. Let $\tilde{\Lambda}$ be the numerical semigroup generated by $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{c-g-1}$. Then, Λ belongs to the infinite chain $\omega(d, \tilde{\Lambda})$.

b. As we have seen in the previous statement, the semigroup Λ', which is a descendant itself of Λ, must be contained in each descendant of Λ.

c. If d is not prime, then $d = d_1 d_2$ for some $d_1, d_2 > 1$ and, as before,

$$\lambda_0 = d_1 d_2 \tilde{\lambda}_0, \lambda_1 = d_1 d_2 \tilde{\lambda}_1, \ldots, \lambda_{c-g-1} = d_1 d_2 \tilde{\lambda}_{c-g-1}$$

with $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{c-g-1}$ coprime. Let $\tilde{\Lambda}$ be $\mathbb{N}_0$ if $c - g - 1 = 1$ or the numerical semigroup generated by $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{c-g-1}$, otherwise. For each $a \geqslant 0$ define

$$\Lambda_a = d_2\tilde{\Lambda} \cup \{\ell \in \mathbb{N}_0 : \ell \geqslant a\}.$$

Then Λ belongs to the infinite chains $\omega(d_1, \Lambda_a)$ for all $a > \frac{c}{d_1}$ that are not multiples of d_2, which are all different. So, it belongs to an infinite number of infinite chains.

d. Suppose that d is prime and that an infinite chain $(\Lambda_i)_{i\geqslant 0}$ contains Λ. The infinite chain must satisfy $\bigcap_{i\geqslant 0} \Lambda_i = d \cdot \tilde{\Lambda}$ for a numerical semigroup $\tilde{\Lambda}$ such that

- $d\tilde{\lambda}_0 = \lambda_0, d\tilde{\lambda}_1 = \lambda_1, \ldots d\tilde{\lambda}_{c-g-1} = \lambda_{c-g-1}$,

- $d\tilde{\lambda}_{c-g} \geqslant c$, since $d\tilde{\Lambda} \subseteq \Lambda$.

Now, the conductor $\tilde{c}$ of $\tilde{\Lambda}$ must be $\tilde{c} \geqslant \tilde{\lambda}_{c-g} \geqslant \lceil \frac{c}{d} \rceil$. Thus, $\tilde{\Lambda}$ is either Λ/d or one of its descendants beyond $\frac{\lambda_{c-g-1}}{d}$.

Conversely, if $\tilde{\Lambda}$ is either Λ/d or one of its descendants beyond $\frac{\lambda_{c-g-1}}{d}$, then Λ belongs to the infinite chain $\omega(d, \tilde{\Lambda})$.

It remains to prove that Λ/d has a finite number of descendants beyond $\frac{\lambda_{c-g-1}}{d}$. If $c-g-1=1$, then $d=\lambda_1 \neq 1$ and $\frac{\lambda_{c-g-1}}{d}=1$, and the semigroup Λ/d is $\mathbb{N}_0$, which has no descendants beyond 1. On the other hand, if $c-g-1 \geqslant 2$, then any descendant beyond $\frac{\lambda_{c-g-1}}{d}$ contains $\frac{\lambda_1}{d}, \ldots, \frac{\lambda_{c-g-1}}{d}$ which is a set of at least two elements which are coprime. Hence, any such descendant contains the numerical semigroup generated by $\frac{\lambda_1}{d}, \ldots, \frac{\lambda_{c-g-1}}{d}$. Now, the result follows from the fact that the number of semigroups that contain a given numerical semigroup is finite.

□

3.1 Minority of semigroups in infinite chains

In this section we prove that for each fixed genus $g \geqslant 5$ the majority of semigroups do not belong to any infinite chain.

From the characterization given by Theorem 3.5 of the semigroups that are in infinite chains, we are able to investigate which children of such semigroups remain in infinite chains.

Corollary 3.6. *If the numerical semigroup Λ lies in an infinite chain, then it has at most two children in infinite chains.*

Proof. Let Λ be a numerical semigroup belonging to an infinite chain, c its conductor, $\mathcal{L}(\Lambda)$ its set of left elements and $d = \gcd(\mathcal{L}(\Lambda))$. By Theorem 3.5, $d \neq 1$ and by definition an effective generator of Λ is $c+i$ for some $i \geqslant 0$. So every child

of Λ is of the type $\Lambda \setminus \{c+i\}$ for some $i \geqslant 0$ and the left elements of this child are either $\mathcal{L}(\Lambda \setminus \{c\}) = \mathcal{L}(\Lambda)$, if $i = 0$, or $\mathcal{L}(\Lambda \setminus \{c+i\}) = \mathcal{L}(\Lambda) \cup \{c, \ldots, c+i-1\}$, if $i > 0$. Now, suppose $i \geqslant 2$. This implies $\mathcal{L}(\Lambda \setminus \{c+i\}) \supseteq \{c, c+1\}$ and, since $\gcd(c, c+1) = 1$, we have $\gcd(\mathcal{L}(\Lambda \setminus \{c+i\})) = 1$. So, $\Lambda \setminus \{c+i\}$, for $i \geqslant 2$, does not have infinitely many descendants. □

Remark 3.7. The proof above makes it clear that the only children of a semigroup Λ that may have infinitely many descendants are $\Lambda \setminus \{c(\Lambda)\}$ and $\Lambda \setminus \{c(\Lambda)+1\}$.

Proposition 3.8. *Except for $\mathbb{N}_0$ and hyperelliptic semigroups, every numerical semigroup that is in an infinite chain has at least one child that is not.*

Proof. Let Λ be a numerical semigroup such that $\gcd(\mathcal{L}(\Lambda)) \neq 1$. Suppose that Λ only has children that are in infinite chains. Let $c = c(\Lambda)$. By Corollary 3.6, the set of effective generators is either $\{c\}$, $\{c+1\}$, or $\{c, c+1\}$.

In the first case, the only child of Λ is $\Lambda_1 = \Lambda \setminus \{c\}$ and, since Λ is in an infinite chain, Λ_1 cannot be a leaf. By Lemma 2.23, Λ_1 has a (unique) effective generator which is $c+m$, where $m = m(\Lambda)$. So, the child of Λ_1 is $\Lambda_2 = \Lambda_1 \setminus \{c+m\} = \Lambda \setminus \{c, c+m\}$. If $m > 2$ then, as c is the conductor of Λ, we have $\{c+1, c+2\} \subset \Lambda \setminus \{c, c+m\} = \Lambda_2$. However, since Λ_2 is the only child of the only child of Λ, it must also be in an infinite chain. That is, $\gcd(\mathcal{L}(\Lambda_2)) \neq 1$ but $\gcd(c+1, c+2) = 1$, a contradiction.

Similarly, in the second case if the only effective generator of Λ is $c+1$, then all its descendants are sticks, the next ones being $\Lambda_1 = \Lambda \setminus \{c+1\}$ and $\Lambda_2 = \Lambda_1 \setminus \{c+1+m\} = \Lambda \setminus \{c+1, c+1+m\}$. And if $m > 2$, $\{c+2, c+3\} \subset \Lambda_2$, contradicting $\gcd(\mathcal{L}(\Lambda_2)) \neq 1$.

Finally, for the third case we assume that c and $c+1$ are the only effective generators of Λ. So, when we take out $c+1$ we are in the same situation as in the second case, where we need $m \leqslant 2$.

Therefore the only possibility for Λ to have all children in infinite chains is if the multiplicity of Λ is equal to 1 or 2, that is, if Λ is $\mathbb{N}_0$ or a hyperelliptic semigroup. □

A direct conclusion from Proposition 3.8 is that, with the exception of $\mathbb{N}_0$ and hyperelliptic semigroups, a numerical semigroup that is in an infinite chain has at least two children.

The way the tree of numerical semigroups is constructed generates subtrees of numerical semigroups with the same multiplicity. We will comment more on this in the next section. For now, consider the following result.

Lemma 3.9. *For every pair (g, m), $2 \leqslant m \leqslant g+1$, there exists a numerical semigroup Λ with $g(\Lambda) = g$ that lies in an infinite chain formed by numerical semigroups of multiplicity m.*

Proof. Consider the chain $\omega(m, \mathbb{N}_0)$ in the notation of Lemma 3.2 and note that, starting from $\{0, m, \rightarrow\}$, all semigroups in this chain have multiplicity m, and since they form an unbroken path in the tree, they achieve all genera $g \geqslant m-1$. □

Lemma 3.10. *Let Λ be a numerical semigroup with enumeration λ, genus g, conductor c, and $\gcd(\mathcal{L}(\Lambda)) = d > 1$. Then the number of children of Λ is at least $d-1$ and if $\lambda_1 = d$, then the number of children of Λ is exactly $d-1$.*

Proof. Since d is a divisor of the left elements of Λ, we have $d \leqslant \lambda_1$, so $d-1 < \lambda_1$. Since $d \neq 1$, the set $\{c, c+1, \ldots, c+d-1\}$ has more than one element and it has exactly one multiple of d. A non-multiple of d of the form $c+i$ with $0 \leqslant i \leqslant d-1$ cannot be generated by $\{\lambda_1, \ldots, \lambda_{c-g-1}\} \cup \{c, c+1, \ldots, c+i-1\}$. Indeed, let $\mathcal{R} = \{c, c+1, \ldots, c+i-1\}$. Suppose that $c+i = \lambda_{j_1} + \cdots + \lambda_{j_k}$. But none of the λ_{j_s} is in $\mathcal{R}$ since, otherwise, if $\lambda_{j_s} \in \mathcal{R}$, then $\lambda_{j_s} \geqslant c$, and so $c+i-\lambda_{j_s} \leqslant i \leqslant d-1 < \lambda_1$, a contradiction. Therefore, every numerical semigroup that is in an infinite chain

has at least $d-1$ children. Moreover, if $\lambda_1 = d$, then, by Proposition 2.18, Λ has exactly $d-1$ effective generators. □

Definition 3.11. A numerical semigroup is *fertile* if most of its children are in infinite chains.

Proposition 3.12. *A numerical semigroup Λ is fertile if and only if one of the options below holds:*

(i) $\Lambda = \mathbb{N}_0$;

(ii) Λ *is a hyperelliptic semigroup;*

(iii) $h(\Lambda) = 3$ *and two of its children are in infinite chains.*

Proof. Let Λ be a fertile numerical semigroup. By Corollary 3.6, if $h(\Lambda) > 3$ then it has at most 2 children in infinite chains, so it cannot be fertile. If $h(\Lambda) = 3$, then it is fertile if it has 2 children in infinite chains, which corresponds to the case (iii). If $h(\Lambda) = 2$, then Λ is fertile if both its children are in infinite chains and, by Proposition 3.8, that occurs when $\Lambda = \mathbb{N}_0 \setminus \{1\}$, which is hyperelliptic as desired. Finally, if $h(\Lambda) = 1$ and its only child is in an infinite chain, then Λ is a hyperelliptic semigroup or $\mathbb{N}_0$. □

There is one hyperelliptic semigroup at each level and we would like to explore what is the frequency of numerical semigroups of type (iii) at each level of the semigroup tree. Consider the semigroup sequence defined by $M_n := \{4k : k \geqslant 0\} \cup [4n+2, \infty)$.

Note that, if $\Lambda = M_n$ for some $n \geqslant 1$, then $c(\Lambda) = 4n+2$ is a minimal generator of Λ, $\gcd(\mathcal{L}(\Lambda)) = \gcd(0, 4, \ldots, 4n) = 4$ and $\gcd(\mathcal{L}(\Lambda \setminus \{c(\Lambda)+1\})) = \gcd(\mathcal{L}(\Lambda), 4n+2) = 2$. So, by Theorem 3.5, $\Lambda \setminus \{c\}$ and $\Lambda \setminus \{c+1\}$ are in infinite chains. In addition, $c(\Lambda) + 3 = 4n+5$ is also an effective generator and, by Proposition 2.18, it is the last of them. Therefore, $\Lambda = M_n$ is of type (iii).

Proposition 3.13. *A numerical semigroup Λ of genus $g > 2$ is of type (iii) if and only if $\Lambda = M_n$ for some $n \geqslant 1$, where $M_n = \{4k : k \geqslant 0\} \cup [4n+2, \infty)$.*

Proof. The only ordinary numerical semigroup which is of type (iii) has genus 2 since the number of effective generators of the ordinary semigroup of genus g is $g+1$, while type (iii) semigroups have efficacy equal to 3. Let Λ be a numerical semigroup of type (iii), and let $g = g(\Lambda) > 2, c = c(\Lambda)$ be its genus and conductor and let λ be its enumeration. Then, $\gcd(\lambda_0, \ldots, \lambda_{c-g-1}) = d > 1$ and, since $h(\Lambda) = 3$, by Lemma 3.10, we have $d \leqslant 4$. Moreover, by Remark 3.7, c and $c+1$ are generators of Λ, with $\gcd(\lambda_0, \ldots, \lambda_{c-g-1}, \lambda_{c-g}) = d' > 1$.

If $d = 2$, then $d' = 2$, c is even, and $\lambda_1 \neq 2$ by Lemma 3.10. In this case, all non-gaps before $c+1$ are even, so if $c+3$ is not a generator, $2 \in \Lambda$, which is a contradiction. Thus $c+3$ is the largest effective generator and the only possibility to write $c+5$ as the sum of non-gaps is $c+5 = (c+1)+4$. Then, $4 \in \Lambda$ and c must be congruent to 2 modulo 4, in order to be a generator. Now, since $d \neq 4$, there must exist $\lambda_i \in \mathcal{L}(\Lambda)$ such that $4 \nmid \lambda_i$. That is, λ_i is congruent to 2 modulo 4. But then $c - \lambda_i$ is a multiple of 4, and so it belongs to Λ, contradicting the fact that c is a generator.

If $d = 3$, similarly we have $3 \mid c$, and so $3 \nmid c+2$. Thus, $c+2$ is not equal to the sum of any elements of Λ smaller than c, so $c+2$ is the third effective generator. Consequently $c, c+1, c+2$ are the three effective generators of Λ and, since $c+4$ is not multiple of 3 and it is not a generator, writing $c+4$ as the sum of non-gaps we get $c+4 = (c+1)+3$ or $c+4 = (c+2)+2$. So, either $2 \in \Lambda$ or $3 \in \Lambda$ but $2 \in \Lambda$ contradicts $d = 3$ and $3 \in \Lambda$ contradicts c being a generator.

If $d = 4$ and $m > 4$, then c is even because $d' \mid 4$ and $d' \neq 1$. Thus, $c+3$ and $c+5$ are odd, and so they are generators of Λ, contradicting $h(\Lambda) = 3$.

If $d = 4$ and $m = 4$, then all left elements are multiples of 4, and c needs to be even and not multiple of 4 since we have $d' \neq 1$ and c a generator. This

description corresponds exactly to the semigroups of the sequence M_n. □

It is easy to see that $g(M_n) = 3n + 1$. Thus we obtain the following result.

Theorem 3.14. *The unique fertile semigroups of genus g are:*

a. $\mathbb{N}_0$ *if* $g = 0$*;*

b. $\mathbb{N}_0 \setminus \{1\}$ *if* $g = 1$*;*

c. $\mathcal{H}_g$ *if* $g > 1$ *and* $g \not\equiv 1 \mod 3$*;*

d. $\mathcal{H}_g$ *and* $M_{\frac{g-1}{3}}$ *if* $g > 1$ *and* $g \equiv 1 \mod 3$.

Proof. Direct from Propositions 3.12 and 3.13. □

Let i_g be the number of numerical semigroups of genus g that are in an infinite chain.

Theorem 3.15. *For* $g \geqslant 5$ *we have* $i_g \leqslant \dfrac{1}{2} n_g$.

Proof. Computations show that $i_5 = 6$ and $n_5 = 12$. For fixed $g \geqslant 6$, let us prove that there are more numerical semigroups with genus g that are not in infinite chains than the other way around. To do so, we partition the set of numerical semigroups of genus g into three disjoint sets:

$$\begin{aligned} F_g &:= \{\Lambda : g(\Lambda) = g \text{ and } \Lambda \cup \{F(\Lambda)\} \text{ is fertile}\} \\ O_g &:= \{\Lambda : g(\Lambda) = g \text{ and } \Lambda \cup \{F(\Lambda)\} \text{ is ordinary}\} \\ P_g &:= \{\Lambda : g(\Lambda) = g \text{ and } \Lambda \notin F_g \sqcup O_g\} \end{aligned}$$

By Lemma 3.9, $P_g \neq \emptyset$. By Theorem 3.14 at most half the nodes in P_g lie in an infinite chain.

If $g-1 \equiv 1 \mod 3$ then, by Theorem 3.14, there exist 3 numerical semigroups in F_g which are in an infinite chain and 1 which is not. In O_g there are 2 elements in an infinite chain and $g-2 \geqslant 4$ that are not. In this way, the difference between the number of elements of $F_g \sqcup O_g$ which are not in an infinite chain and the number of those that are is non-negative as well. If $g-1 \not\equiv 1 \mod 3$, then F_g has only 1 element and it belongs to an infinite chain and the result holds similarly. □

3.2 Fixing the multiplicity

In the tree that organizes all semigroups by genus, the set of descendants of $\mathbb{N}_0 \setminus \{1,3\}$ is a single infinite chain of hyperelliptic semigroups. Indeed, $\mathbb{N}_0 \setminus \{1,3\}$ is the third semigroup in $I_{\mathcal{H}}$. On the other hand $\mathbb{N}_0 \setminus \{1,2\}$ is in $I_{\mathcal{O}}$, the infinite chain formed by the ordinary semigroups. From each ordinary semigroup in $I_{\mathcal{O}}$ there emerges a subtree that contains all the semigroups of the same multiplicity (as one can see in [15]). From now on, we focus on these subtrees with fixed multiplicity. One way to transit in such a subtree is, from a semigroup in it, to move to another numerical semigroup of higher genus while maintaining the structure of the first semigroup, that is, maintaining the structure of its non-gaps. The construction explained in next definition does this: it pushes the non-gaps of a numerical semigroup while keeping its multiplicity.

Definition 3.16. The *push* of a numerical semigroup Λ, with enumeration λ, *by its multiplicity* is $\lambda_1 \oplus \Lambda := \{0\} \cup \{\lambda_1 + \lambda_j : \lambda_j \in \Lambda\}$.

The set $\lambda_1 \oplus \Lambda$ as above is a numerical semigroup. In fact, if $x, y \in \lambda_1 \oplus \Lambda$ then, either x, y or both are zero, in which case the sum belongs to $\lambda_1 \oplus \Lambda$, or there exist $\lambda_a, \lambda_b \in \Lambda$ such that $x = \lambda_1 + \lambda_a$ and $y = \lambda_1 + \lambda_b$. In this case, $x + y = \lambda_1 + (\lambda_a + \lambda_1 + \lambda_b) \in \lambda_1 \oplus \Lambda$. In addition, if $x \geqslant \lambda_1 + c(\Lambda)$, then $x - \lambda_1 \in \Lambda$, and so $x \in \lambda_1 \oplus \Lambda$. That is, $\lambda_1 + c(\Lambda) - 1$ is the maximum of $(\lambda_1 \oplus \Lambda)^C$ and, thus, $(\lambda_1 \oplus \Lambda)^C$ is finite. Moreover, $m(\lambda_1 \oplus \Lambda) = \lambda_1$, $g(\lambda_1 \oplus \Lambda) = g(\Lambda) + \lambda_1 - 1$,

$c(\lambda_1 \oplus \Lambda) = \lambda_1 + c(\Lambda)$, $\mathcal{L}(\lambda_1 \oplus \Lambda) = \{0\} \cup \{\lambda_1 + \lambda_j; \lambda_j \in \mathcal{L}(\Lambda)\}$ and $\#\mathcal{L}(\lambda_1 \oplus \Lambda) = \#\mathcal{L}(\Lambda) + 1$.

Lemma 3.17. *Let Λ be a numerical semigroup with enumeration λ and let Π be a non-ordinary numerical semigroup. Then,*

a. If λ_k is an (effective) generator of Λ, then $\lambda_1 + \lambda_k$ is an (effective) generator of $\lambda_1 \oplus \Lambda$.

b. Π is a child of Λ if and only if $\lambda_1 \oplus \Pi$ is a child of $\lambda_1 \oplus \Lambda$.

c. Λ lies in an infinite chain if and only if so does $\lambda_1 \oplus \Lambda$.

Proof. Direct by Definition 3.16. □

The reciprocal of implication a is not true in general. For instance, $\langle 4, 5, 6\rangle$ is a leaf, while $4 \oplus \langle 4, 5, 6\rangle$ is a stick. From now on, we use the notation $\lambda_1 \oplus^n \Lambda$ to represent $\underbrace{\lambda_1 \oplus (\cdots (\lambda_1 \oplus (\lambda_1 \oplus \Lambda)))}_{n \text{ times}}$.

Now we will investigate infinite chains in subtrees with some fixed multiplicity. We call *infinite chains of multiplicity m* those infinite chains whose first m semigroups are the ordinary semigroups of multiplicities 1 to m and after that all semigroups in the chain have multiplicity m. Equivalently, an infinite chain is an infinite chain of multiplicity m if and only if the maximum of the multiplicities of the semigroups in the chain is m.

3.2.1 Prime multiplicity

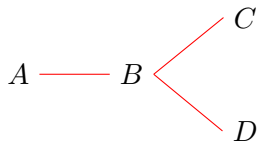
From Theorem 3.5 one can deduce that, for each prime multiplicity, there is only one infinite chain of that multiplicity. Indeed, this unique chain is $\omega(m, \mathbb{N}_0)$. Hence, we can state the following theorem.

Theorem 3.18. *If m is a prime, then the number $i_g(m)$ of numerical semigroups of genus $g \geqslant m - 1$ and multiplicity m that are in an infinite chain is 1.*

3.2.2 Multiplicity 4

The tree in Figure 3.1 represents the numerical semigroups of multiplicity 4, genus from 3 to 5 and which are in infinite chains. The edges connect semigroups as usual, being a child of a given semigroup the given semigroup minus one of its effective generators.

Figure 3.1: Tree structure of numerical semigroups of multiplicity 4 that are in an infinite chain, from genus 3 to 5.



Specifically: $A = \{0, 4, \rightarrow\}, B = \{0, 4, 6, \rightarrow\}, C = \{0, 4, 7, \rightarrow\}$, and $D = \{0, 4, 6, 8, \rightarrow\}$.

As we can observe, $c(C)$ is a generator while $c(C) + 1$ is not. So by Remark 3.7 C has only one child in an infinite chain and this child is exactly $4 \oplus A$. In general, for $n \geqslant 1$, $4 \oplus^n C \setminus \{c(4 \oplus^n C)\} = 4 \oplus^{n+1} A$ is the only child of $4 \oplus^n C$ that is in an infinite chain. Moreover, $4 \oplus^n A$ has one child in an infinite chain, and so by Lemma 3.17 this child is $4 \oplus^n B$.

On the other hand, the efficacy of $4 \oplus^n D = \{0, 4, \cdots, 4n, 4n + 2, 4n + 4, \rightarrow\}$, with $n \geqslant 1$, is equal to 2. So the efficacy of any descendant of $4 \oplus^n D$ is at most 2. Then by Proposition 3.8 every descendant of $4 \oplus^n D$ has exactly one child in an infinite chain. So, the infinite chain emerging from a node of the form $4 \oplus^n D$ is made of sticks in the tree of semigroups that are in infinite chains. In that way we have the following theorem.

Theorem 3.19. *The number $i_g(4)$ of numerical semigroups of genus $g \geqslant 3$ and*

multiplicity 4 *that are in an infinite chain is*

$$i_g(4) = \left\lfloor \frac{g+1}{3} \right\rfloor.$$

Proof. It is clear for $g = 3, 4$. Otherwise, the tree in Figure 3.1 is self-replicated in the tree of infinite chains of multiplicity 4 starting at $\{0, 4, \rightarrow\}$. Moreover, as observed before, the numerical semigroups of multiplicity 4 that are in an infinite chain and do not participate in self-replication are sticks in the tree of semigroups that are in infinite chains. □

Example 3.20. In the tree in Figure 3.2 we can observe thirteen complete self-replications of the tree in Figure 3.1. The figure was generated with the `drawsgtree` tool [20], using the command

```
./drawsgtree -g41 -m4 -e infinitechains -etrim -d .2
```

Figure 3.2: Tree of numerical semigroups in infinite chains, with multiplicity 4, from genus 3 up to 41.



3.2.3 Multiplicity 6

The numerical semigroup of multiplicity 6 and with the smallest genus is the ordinary semigroup $\{0, 6, \rightarrow\}$. It has two children in infinite chains: the ordinary semigroup $\{0, 7, \rightarrow\}$ and the semigroup $\{0, 6, 8, \rightarrow\}$. To study the numerical semigroups that are in infinite chains and have multiplicity 6, we study the tree v formed by $\{0, 6, \rightarrow\}$ in its root and connected to $\{0, 6, 8, \rightarrow\}$ and all semigroups descending from it that are in infinite chains. In other words, v is formed by

the subtree rooted at $\{0, 6, \rightarrow\}$ that contains all numerical semigroups in infinite chains, where the node $\{0, 7, \rightarrow\}$ and all its descendants have been trimmed.

The semigroups of υ are of type $\Lambda_n = \{0, 6, \ldots, 6n, 6n + \beta_1, \ldots, 6n + \beta_j, \rightarrow\}$. In this case we can use the j-tuple $(\beta_1, \ldots, \beta_j)_n$ to represent Λ_n. For example, we represent $\{0, 6, 8, \rightarrow\}$ by $(2)_1$ and $\{0, 6, 12, 14, 16, 18, 20, 22, 24, \rightarrow\}$ by $(2, 4, 6, 8, 10, 12)_2$. We use $(\cdot)_n$ to represent the ordinary semigroup of genus $6n-1$.

The tree τ_n and its replications

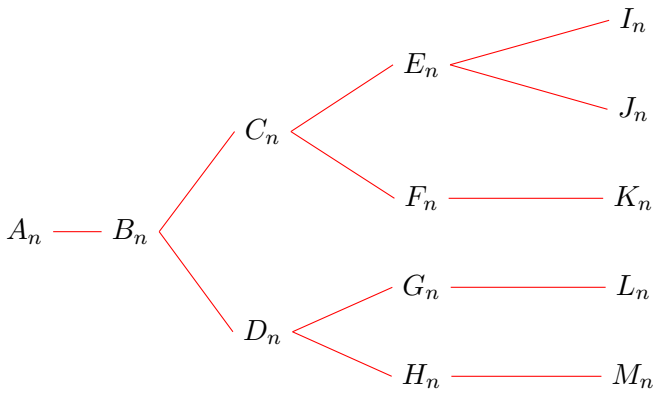
In Table 3.1 we have thirteen numerical semigroups that have multiplicity 6, are in infinite chains and have some parenting relationships that can be observed in the tree τ_n in Figure 3.3. By Remark 3.7 and the last column in Table 3.1, we verify that each semigroup Λ_n in τ_n has children in infinite chains as drawn in Figure 3.1.

Table 3.1: Some numerical semigroups that have multiplicity 6 and are in infinite chains.

Label	Numerical Semigroup	Only children
A_n	$(\cdot)_n$	$c(A_n) = 6n$ is not a minimal generator
B_n	$(2)_n$	
C_n	$(3)_n$	
D_n	$(2, 4)_n$	
E_n	$(4)_n$	
F_n	$(3, 5)_n$	$c(F_n) + 1 = 6n + 6$ is not a minimal generator
G_n	$(2, 5)_n$	$c(G_n) + 1 = 6n + 6$ is not a minimal generator
H_n	$(2, 4, 6)_n$	$c(H_n) = 6n + 6$ is not a minimal generator
I_n	$(5)_n$	$c(I_n) + 1 = 6n + 6$ is not a minimal generator
J_n	$(4, 6)_n$	
K_n	$(3, 6)_n$	
L_n	$(2, 6)_n$	
M_n	$(2, 4, 6, 8)_n$	

Note that I_n has only one child in an infinite chain which is $I_n \backslash \{c(I_n)\} = 6 \oplus A_n$. Moreover, $6 \oplus \Lambda_n = \Lambda_{n+1}$ for all $n \geqslant 1$ and Λ_n a numerical semigroup from Table 3.1. Therefore, the finite tree in Figure 3.3 is self-replicated in υ through push by multiplicity.

Figure 3.3: Finite tree τ_n.



Outside the replications of τ_n

Now we focus on the numerical semigroups of multiplicity 6 that are in an infinite chain and do not participate in that self-replication, i.e., the descendants of J_n, K_n, L_n, and M_n.

Proposition 3.21. *The numerical semigroups* $K_n = \{0, 6, \ldots, 6n, 6n+3, 6n+6, \rightarrow\}$ *and* $M_n = \{0, 6, \ldots, 6n, 6n+2, 6n+4, 6n+6, 6n+8, \rightarrow\}$ *each belong to one infinite chain.*

Proof. By Theorem 3.5 the number of infinite chains to which K_n belongs is one plus the number of descendants of $\tilde{K}_n = \{0, 2, \ldots, 2n, 2n+1, 2n+2, \rightarrow\}$ beyond $2n+1$. But $\tilde{K}_n$ has no generators larger than $2n+1$, and so it has no descendants beyond $2n+1$. Thus, K_n only belongs to one infinite chain.

Similarly, the number of infinite chains to which M_n belongs is one plus the number of descendants of $\tilde{M}_n = \{0, 3, \ldots, 3n, 3n+1, 3n+2, 3n+3, 3n+4, \rightarrow\}$ beyond $3n+3$. But $\tilde{M}_n$ has no generators larger than $3n+3$, and so it has no descendants beyond $3n+3$. Thus, M_n only belongs to one infinite chain. □

Proposition 3.22. *The numerical semigroups*

$$J_n = \{0, 6, \ldots, 6n, 6n+4, 6n+6, \rightarrow\}$$

belongs to $n+1$ infinite chains.

Proof. By Theorem 3.5, the number of infinite chains in which J_n lies is one plus the number of descendants of $\tilde{J}_n = \{0, 3, \ldots, 3n, 3n+2, 3n+3, \rightarrow\}$ beyond $3n+2$.

If $n = 1$, $\tilde{J}_n$ has one descendant beyond 5. Hence, J_1 lies in two infinite chains.

Suppose $n > 1$. Observe that $\tilde{J}_n$ has one unique child that contains $\{0, 3, \ldots, 3n, 3n+2\}$, which is $\tilde{J}_n \setminus \{3n+4\}$. Observe also that any semigroup containing $\{0, 3, \ldots, 3n, 3n+2\}$ must contain the semigroup

$$\{0, 3, \ldots, 3n, 3n+1, 3(n+1), 3(n+1)+2 \ldots, 3(2n), 3(2n)+2, \rightarrow\}.$$

One can deduce that all descendants of $\tilde{J}_n$ beyond $3n+2$ are exactly the semigroups $\tilde{J}_{n,t} = \{0, 3, \ldots, 3n, 3n+2, 3(n+1), 3(n+1)+2, \ldots, 3(n+t), 3(n+t)+2, \rightarrow\}$ for $2 \leqslant t \leqslant n+1$. Hence, J_n lies in exactly $n+1$ infinite chains. □

Proposition 3.23. *The numerical semigroups*

$$L_n = \{0, 6, \ldots, 6n, 6n+2, 6n+6, \rightarrow\}$$

belongs to n infinite chains.

Proof. By Theorem 3.5, the number of infinite chains in which L_n lies is one plus the number of descendants of $\tilde{L}_n = \{0, 3, \ldots, 3n, 3n+1, 3n+3, \rightarrow\}$ beyond $3n+1$.

If $n = 1$, $\tilde{L}_n$ has no descendants beyond 4. Hence, L_1 lies in exactly one infinite chain.

Suppose $n > 1$. Observe that $\tilde{L}_n$ has one unique child, which is $\tilde{L}_n \setminus \{3n+4\}$. Observe also that any semigroup containing $\{0, 3, \ldots, 3n, 3n+1\}$ must contain the semigroup

$$\{0, 3, \ldots, 3n, 3n+1, 3(n+1), 3(n+1)+1 \ldots, 3(2n), 3(2n)+1, 3(2n)+2, \rightarrow\}.$$

One can deduce that all descendants of $\tilde{L}_n$ beyond $3n+1$ are exactly the semigroups $\tilde{L}_{n,t} = \{0, 3, \ldots, 3n, 3n+1, 3(n+1), 3(n+1)+1, \ldots, 3(n+t), 3(n+t)+1, \rightarrow\}$ for $2 \leqslant t \leqslant n$. Hence, L_n lies in exactly n infinite chains. □

In order to know at which levels of the tree of numerical semigroups the descendants of the semigroups J_n and L_n has two children in infinite chains, let us define in the next two lemmas some numerical semigroups.

Lemma 3.24. *For every integer t with $1 \leqslant t \leqslant n$, the set*

$$\begin{aligned}\gamma_{n,t} &= \{0, 6, \ldots, 6n, 6n+4, 7n, 7n+4, \ldots, 6(n+t-1), 6(n+t-1)+4, \\ &\qquad 6(n+t), 6(n+t)+2, \rightarrow\} \\ &= \{0, 6, \ldots, 6n, 6n+a_1, 6n+a_2, \ldots, 6n+a_{2t+1}, \rightarrow\},\end{aligned}$$

with

$$a_i = \begin{cases} 3i+1 & \text{if } i \text{ is odd and } i \neq 2t+1 \\ 3i, & \text{if } i \text{ is even} \\ 3i-1, & \text{if } i = 2t+1. \end{cases}$$

satisfies:

a. $\gamma_{n,t}$ *is a numerical semigroup;*

b. $\gamma_{n,1}$ *is a child of* J_n *and, for* $t \geqslant 2$, $\gamma_{n,t}$ *is a descendant of* J_n*;*

c. If $t \leqslant n$, *then* $\gamma_{n,t}$ *has two children in infinite chains;*

d. The semigroups J_n, $\gamma_{n,1}, \gamma_{n,2}, \ldots, \gamma_{n,n}$ *are in the same infinite chain;*

e. The genus of $\gamma_{n,t}$ *is* $5n + 4t + 1$.

The proof is straightforward and is left to the reader.

Let ζ_n be the infinite chain where all the semigroups $\{J_n, \gamma_{n,t}\}_{1 \leqslant t \leqslant n}$ of Lemma 3.24 belong. Since each $\gamma_{n,t}$ has two children in infinite chains, one child belongs to ζ_n and the other one belongs to a different infinite chain. Thus, the $n+1$ infinite chains from Proposition 3.23 to which S_n belongs are ζ_n and n other chains, each of them containing $\gamma_{t,n} \setminus \{c(\gamma_{t,n}) + 1\}$, for t between 1 and n.

Lemma 3.25. *For every integer* t *with* $1 \leqslant t \leqslant n-1$, *the set*

$$\begin{aligned} \nu_{n,t} &= \{0, 6, \ldots, 6n, 6n+2, 7n, 7n+2, \ldots, 6(n+t-1), 6(n+t-1)+2, \\ &\qquad 6(n+t), 6(n+t)+2, 6(n+t)+4 \to\} \\ &= \{0, 6, \ldots, 6n, 6n+b_1, 6n+b_2, \ldots, 6n+b_{2t+2}, \to\}, \end{aligned}$$

with

$$b_i = \begin{cases} 3i-1 & \text{if } i \text{ is odd} \\ 3i, & \text{if } i \text{ is even and} i \neq 2t+2 \\ 3i-2, & \text{if } i = 2t+2. \end{cases}$$

satisfies:

a. $\nu_{n,t}$ *is a numerical semigroup;*

b. $\nu_{n,1}$ *is a grandchild of* L_n *and, for* $t \geqslant 2$, $\nu_{n,t}$ *is a descendant of* L_n*;*

c. *If* $t \leqslant n-1$, *then* $\nu_{n,t}$ *has two children in infinite chains;*

d. *The semigroups* $L_n, \nu_{n,1}, \nu_{n,2}, \ldots, \nu_{n,n-1}$ *are in the same infinite chain;*

e. *The genus of* $\nu_{n,t}$ *is* $5n + 4t + 2$.

Again, the proof is straightforward and is left to the reader.

Let η_n be the infinite chain where all the semigroups $\{L_n, \nu_{n,t}\}_{1 \leqslant t \leqslant n-1}$ of Lemma 3.25 belong. For n fixed, since each $\nu_{n,t}$ have two children in infinite chains, one child belongs to η_n and the other one belongs to a different infinite chain. Thus, the n infinite chains from Proposition 3.23 to which L_n belongs are η_n and $n-1$ other chains, each one containing $\nu_{t,n} \setminus \{c(\nu_{t,n}) + 1\}$, for t between 1 and $n-1$.

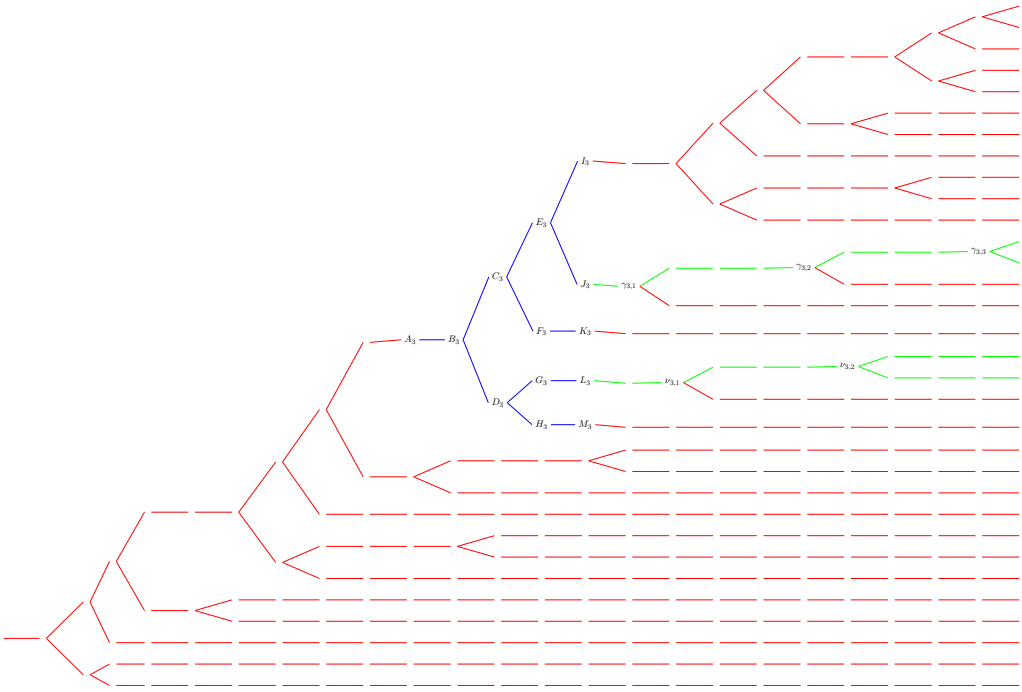
Growth of the tree υ

In Figure 3.4 we drew the edges and nodes of υ corresponding to semigroups from genus 5 up to genus 29. The tree τ_3 and the infinite chains ζ_3 and η_3 are highlighted. Notice also that one can observe five full self-replications of the tree from Figure 3.3. The figure was generated with the `drawsgtree` tool [20] with the command `./drawsgtree -g29 -m6 -e infinitechains -etrim -d .1`.

The number $i_g(6)$ of numerical semigroups of genus g and multiplicity 6 that are in an infinite chain whenever $g \geqslant 5$, is equal to the number of numerical semigroups on level $g-5$ of the tree υ. From what has been developed in Subsection 3.2.3, the growth of the tree υ is known and we are ready to state it in the next theorem.

Lemma 3.26. *The number* $i_g(6)$ *of numerical semigroups of genus* $g \geqslant 5$ *and*

Figure 3.4: Numerical semigroups of multiplicity 6 in infinite chains, from genus 5 up to genus 29. The subtree τ_3 is highlighted in blue. The parts of the chains η_3 and ζ_3 of genus larger than or equal to 19 are highlighted in green.



multiplicity 6 *that are in an infinite chain is*

$$\begin{cases} g-5-\left\lfloor\frac{g-5}{5}\right\rfloor+\sum\limits_{n=1}^{\lfloor\frac{g-2}{9}\rfloor} n+\sum\limits_{n=\lfloor\frac{g+7}{9}\rfloor}^{\lfloor\frac{g-6}{5}\rfloor}\left\lfloor\frac{g-2-5n}{4}\right\rfloor+\sum\limits_{n=2}^{\lfloor\frac{g+1}{9}\rfloor}(n-1)+\sum\limits_{n=\lfloor\frac{g+10}{9}\rfloor}^{\lfloor\frac{g-7}{5}\rfloor}\left\lfloor\frac{g-3-5n}{4}\right\rfloor \\ \text{if } g\equiv 1,2 \mod 5, \\ g-4-\left\lfloor\frac{g-5}{5}\right\rfloor+\sum\limits_{n=1}^{\lfloor\frac{g-2}{9}\rfloor} n+\sum\limits_{n=\lfloor\frac{g+7}{9}\rfloor}^{\lfloor\frac{g-6}{5}\rfloor}\left\lfloor\frac{g-2-5n}{4}\right\rfloor+\sum\limits_{n=2}^{\lfloor\frac{g+1}{9}\rfloor}(n-1)+\sum\limits_{n=\lfloor\frac{g+10}{9}\rfloor}^{\lfloor\frac{g-7}{5}\rfloor}\left\lfloor\frac{g-3-5n}{4}\right\rfloor \\ \text{if } g\equiv 0,3,4 \mod 5. \end{cases}$$

Proof. First of all, notice that the number of self-replications of the structure of τ_1, from Figure 3.3, which lie before level $g-5$ of v is $\lfloor\frac{g-5}{5}\rfloor$.

For fixed g we can define four sets of numerical semigroups:

$$\begin{aligned} A &:= \left\{\Lambda : \Lambda \in \tau_{\lfloor\frac{g-5}{5}\rfloor+1} \text{ and } g(\Lambda)=g\right\}, \\ B &:= \left\{\Lambda : \Lambda \text{ is a descendant of } K_n \text{ or } M_n, 1\leqslant n\leqslant\left\lfloor\frac{g-5}{5}\right\rfloor, \text{ and } g(\Lambda)=g\right\}\cup \\ &\quad \left\{\Lambda : \Lambda \text{ is a descendant of } J_n \text{ or } L_n, 1\leqslant n\leqslant\left\lfloor\frac{g-5}{5}\right\rfloor, \Lambda\in\zeta_n\cup\eta_n, g(\Lambda)=g\right\}, \\ C &:= \left\{\Lambda : \Lambda \text{ is a descendant of } J_n, 1\leqslant n\leqslant\left\lfloor\frac{g-5}{5}\right\rfloor, \Lambda\notin\zeta_n, \text{ and } g(\Lambda)=g\right\}, \\ D &:= \left\{\Lambda : \Lambda \text{ is a descendant of } L_n, 2\leqslant n\leqslant\left\lfloor\frac{g-5}{5}\right\rfloor, \Lambda\notin\eta_n, \text{ and } g(\Lambda)=g\right\}. \end{aligned}$$

In the definition of D the case $n=1$ has been excluded since L_1 belongs only to one infinite chain which is exactly η_1.

Now, the numerical semigroups of genus $g\geqslant 5$ and multiplicity 6 which are in an infinite chain are the numerical semigroups of $A\sqcup B\sqcup C\sqcup D$.

To get the cardinality of A, consider the remainder r such that $g-5=5\lfloor\frac{g-5}{5}\rfloor+r$. It is the level of the tree $\tau_{\lfloor\frac{g-5}{5}\rfloor+1}$ at which the numerical semigroups of genus g that participate in self-replication lie. So, the cardinality of A is equal to r if $r=1,2$ and equal to $r+1$ when $r=0,3,4$.

By Proposition 3.21, Lemma 3.24, and Lemma 3.25, the cardinality of B is equal to $4\lfloor\frac{g-5}{5}\rfloor$.

On the other hand, $5\left\lfloor\frac{g-5}{5}\right\rfloor = g-5-r$. So,

$$\#A+\#B=\begin{cases} g-5-\lfloor\frac{g-5}{5}\rfloor & \text{if } g\equiv 1,2 \mod 5\\ g-4-\lfloor\frac{g-5}{5}\rfloor & \text{if } g\equiv 0,3,4 \mod 5.\end{cases}$$

The descendants of J_n, for $1\leqslant n\leqslant\left\lfloor\frac{g-5}{5}\right\rfloor$, which belong to infinite chains but do not belong to ζ_n and have genus g, $g\geqslant 10$, are in bijection with the numerical semigroups $\gamma_{n,t}$ from Lemma 3.24 such that $g(\gamma_{n,t})<g$. Since $g(\gamma_{n,t})=5n+4t+1$, we have that the cardinality of C is equal to

$$\begin{aligned} & \#\left\{(n,t): 5n+4t+1<g, \text{ where } 1\leqslant n\leqslant\left\lfloor\tfrac{g-5}{5}\right\rfloor \text{ and } 1\leqslant t\leqslant n\right\}\\ = \ & \#\left\{(n,t): 5n+4t+2\leqslant g, \text{ where } 1\leqslant n\leqslant\left\lfloor\tfrac{g-5}{5}\right\rfloor \text{ and } 1\leqslant t\leqslant n\right\}\\ = \ & \sum_{n=1}^{\lfloor\frac{g-2}{9}\rfloor} n+\sum_{n=\lfloor\frac{g+7}{9}\rfloor}^{\lfloor\frac{g-6}{5}\rfloor}\left\lfloor\tfrac{g-2-5n}{4}\right\rfloor.\end{aligned}$$

On the other hand, by Lemma 3.25, the set D is in bijection with the set $\{\nu_{n,t} : 2\leqslant n\leqslant\left\lfloor\frac{g-5}{5}\right\rfloor, 1\leqslant t\leqslant n-1 \text{ and } g(\nu_{n,t})<g\}$. Furthermore, since $g(\nu_{n,t})=5n+4t+2$, we have that the cardinality of D is equal to

$$\begin{aligned} & \#\left\{(n,t): 5n+4t+2<g, \text{ where } 2\leqslant n\leqslant\left\lfloor\tfrac{g-5}{5}\right\rfloor \text{ and } 1\leqslant t\leqslant n-1\right\}\\ = \ & \#\left\{(n,t): 5n+4t+3\leqslant g, \text{ where } 2\leqslant n\leqslant\left\lfloor\tfrac{g-5}{5}\right\rfloor \text{ and } 1\leqslant t\leqslant n-1\right\}\\ = \ & \sum_{n=2}^{\lfloor\frac{g+1}{9}\rfloor}(n-1)+\sum_{n=\lfloor\frac{g+10}{9}\rfloor}^{\lfloor\frac{g-7}{5}\rfloor}\left\lfloor\tfrac{g-3-5n}{4}\right\rfloor\\ = \ & \sum_{n=1}^{\lfloor\frac{g-8}{9}\rfloor} n+\sum_{n=\lfloor\frac{g+10}{9}\rfloor}^{\lfloor\frac{g-7}{5}\rfloor}\left\lfloor\tfrac{g-3-5n}{4}\right\rfloor.\end{aligned}$$

□

The formula given in Lemma 3.26 is nothing else but $i_g(6) = \left\lfloor \frac{2g^2+g-16}{9} \right\rfloor - \left\lfloor \frac{(g-1)^2}{5} \right\rfloor$. It can be proved by some arguments on quasi-polynomials or using the implementation of Barvinok's algorithm in LattE [30]. Hence, we can state now the following theorem.

Theorem 3.27. *The number $i_g(6)$ of numerical semigroups of genus $g \geqslant 5$ and multiplicity 6 that are in an infinite chain is*

$$i_g(6) = \left\lfloor \frac{2g^2+g-16}{9} \right\rfloor - \left\lfloor \frac{(g-1)^2}{5} \right\rfloor.$$

3.3 Conjecture and its roots in prior work

To derive the formulas for the number of numerical semigroups of genus g and multiplicity m that are in an infinite chain, $i_g(m)$, for the cases $m = 4, 6$, a crucial tool was the concept of push of a numerical semigroup by its multiplicity. The way it was defined, these pushes correspond to maximal embedding dimension (MED) semigroups [5, Proposition I.2.9]. Let $N(m, g)$ denote the number of numerical semigroups of genus g and multiplicity m, and let $\mathrm{MED}(m, g)$ represent the number of MED semigroups with genus g and multiplicity m. In this discussion, we bring forth several outcomes concerning these numbers and their relationship with $i_g(m)$.

Rosales [47] was the first to compute the value of $N(m, g)$ for $m = 3$. In their 2012 preprint [39], García-Sánchez, Marín-Aragón, and Robles-Pérez provide extensive computations using the Kunz polytope and tree patterns to determine $N(m, g)$ for m up to 5. In 2012, Kaplan [41, Proposition 7] demonstrated that for a fixed $m \geqslant 3$ and $g > 0$, both $N(m, g)$ and $\mathrm{MED}(m, g)$ are eventually quasi-polynomials of degree $m-2$, with a period that depends on m. Later, in 2019, Alhajjar, Russell, and Steward [3] proved that the leading coefficients of $N(m, g)$ and $\mathrm{MED}(m, g)$ are constant and share the same value. Utilizing the quasi-polynomial characterization, [3] subsequently established that, asymptotically in g, almost all

numerical semigroups of genus g and fixed multiplicity m have maximal embedding dimension. Formula $i_g(m)$ of Theorem 3.27 counts a subset of the elements considered by $N(m, g)$. Our discoveries suggest the following conjecture.

Conjecture 3.28. *For a fixed multiplicity m, the number of semigroups of genus g lying in an infinite chain is quasi-polynomial in g, for $g \geqslant m - 1$.*

Additionally, we underline two more significant contributions from previous studies on counting numerical semigroup that corroborate the quasi-polynomial direction of Conjecture 3.28. The 2019 result of Alhajjar, Russell, and Steward [3] for $N(4, g)$ aligns with that obtained by García-Sánchez, Marín-Aragón, and Robles-Pérez [39] in 2018, albeit expressed differently: while [39] employed computations involving floor functions, [3] utilized quasi-polynomials. As a future direction, we would like to compute $i_g(m)$ for any m, using the pattern offered by push and obtaining quasi-polynomial expressions.

Chapter 4

Rarity of the Infinite Chains

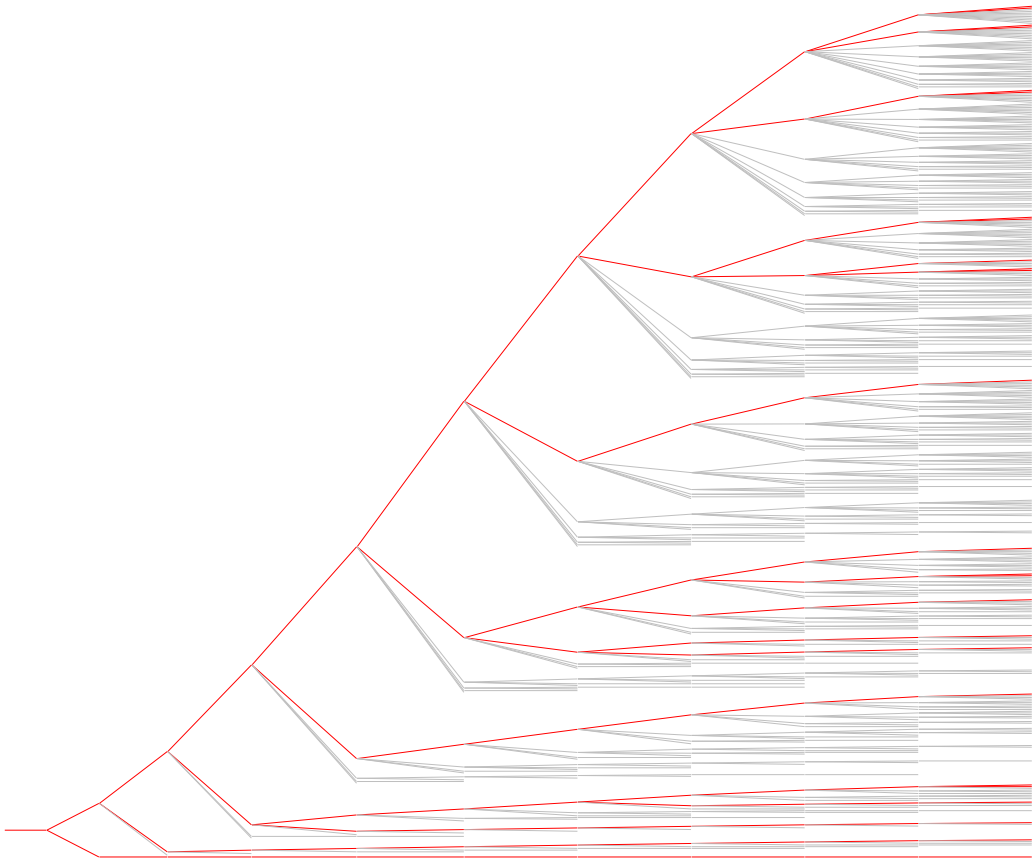
Pequenina e também gigante.

Marisa Monte, Brazilian singer and songwriter, in the song Infinito Particular

Computation shows that semigroups in infinite chains are rare, although the tree grows exponentially. In Figure 4.1 one can see the tree structure of the semigroup tree up to genus 11. The edges corresponding to infinite chains have been highlighted. It can be appreciated that they constitute a very small portion of the whole set of edges up to this genus.

In this chapter we prove that, for each fixed genus, the portion of semigroups of that genus belonging to infinite chains in the semigroup tree approaches 0 as the genus grows to infinity. This means that most numerical semigroups have a finite number of descendants in the semigroup tree. This problem has been open since 2009.

Figure 4.1: Tree structure of the semigroup tree up to genus 11.



In our proof we will use the asymptotic results on the multiplicity and the Frobenius number of a numerical semigroup mentioned at the end of Section 2.1 together with the characterization of semigroups in infinite chains presented in Chapter 3.

Let Λ be a numerical semigroup. Considering the sets $\mathcal{M}_g$ (set in 2.1) and $\mathcal{F}_g$ (set in 2.1), define

$$\begin{aligned}
\mathcal{MF}_g(\varepsilon_1,\varepsilon_2) &= \mathcal{M}_g(\varepsilon_1)\cap\mathcal{F}_g(\varepsilon_2),\\
\overline{S_g^{\infty}} &= \{\Lambda\in S_g : \gcd(\mathcal{L}(\Lambda))=1\},\ \text{and}\\
B_g(\varepsilon) &= \{\Lambda\in S_g : (\gamma-\varepsilon)g<m(\Lambda)<(\gamma+\varepsilon)g\}\cap\\
&\quad \{\Lambda\in S_g : (2-\varepsilon)\gamma g<F(\Lambda)<(2+\varepsilon)\gamma g\},
\end{aligned}$$

Lemma 4.1. *For any fixed pair $\varepsilon_1>0,\varepsilon_2>0$, $\lim_{g\to\infty}\frac{\#\mathcal{MF}_g(\varepsilon_1,\varepsilon_2)}{n_g}=1$.*

Proof. It is a consequence of the fact that

$$\begin{aligned}
S_g &= \mathcal{MF}_g(\varepsilon_1,\varepsilon_2)\sqcup(\mathcal{M}_g(\varepsilon_1)\setminus\mathcal{F}_g(\varepsilon_2))\sqcup(\mathcal{F}_g(\varepsilon_2)\setminus\mathcal{M}_g(\varepsilon_1))\\
&\quad \sqcup(S_g\setminus((\mathcal{M}_g(\varepsilon_1)\cup\mathcal{F}_g(\varepsilon_2)))
\end{aligned}$$

and so,

$$\begin{aligned}
\#\mathcal{MF}_g(\varepsilon_1,\varepsilon_2) &= n_g-\#(\mathcal{M}_g(\varepsilon_1)\setminus\mathcal{F}_g(\varepsilon_2))-\#(\mathcal{F}_g(\varepsilon_2)\setminus\mathcal{M}_g(\varepsilon_1))\\
&\quad -\#(S_g\setminus(\mathcal{M}_g(\varepsilon_1)\cup\mathcal{F}_g(\varepsilon_2))),
\end{aligned}$$

$$\begin{aligned}
\#\mathcal{MF}_g(\varepsilon_1,\varepsilon_2) &= n_g-\#(\mathcal{M}_g(\varepsilon_1)\setminus\mathcal{F}_g(\varepsilon_2))-\#(\mathcal{F}_g(\varepsilon_2)\setminus\mathcal{M}_g(\varepsilon_1))\\
&\quad -\#(S_g\setminus(\mathcal{M}_g(\varepsilon_1)\cup\mathcal{F}_g(\varepsilon_2))),
\end{aligned}$$

and the fact that, by Lemma 2.19 and Lemma 2.20,

$$\begin{array}{rcccl} \lim_{g\to\infty} \frac{\#(\mathcal{M}_g(\varepsilon_1)\setminus\mathcal{F}_g(\varepsilon_2))}{n_g} & \leqslant & \lim_{g\to\infty} \frac{\#(S_g\setminus\mathcal{F}_g(\varepsilon_2))}{n_g} & = & 0, \\ \lim_{g\to\infty} \frac{\#(\mathcal{F}_g(\varepsilon_2)\setminus\mathcal{M}_g(\varepsilon_1))}{n_g} & \leqslant & \lim_{g\to\infty} \frac{\#(S_g\setminus\mathcal{M}_g(\varepsilon_1))}{n_g} & = & 0, \\ \lim_{g\to\infty} \frac{\#(S_g\setminus(\mathcal{M}_g(\varepsilon_1)\cup\mathcal{F}_g(\varepsilon_2)))}{n_g} & \leqslant & \lim_{g\to\infty} \frac{\#(S_g\setminus(\mathcal{M}_g(\varepsilon_1))}{n_g} & = & 0. \end{array}$$

□

Lemma 4.2. *For any fixed $\varepsilon > 0$, setting $\varepsilon_1 = \frac{\gamma\varepsilon}{4+\varepsilon}$ and $\varepsilon_2 = \frac{\varepsilon}{2}$, it holds*

$$\mathcal{MF}_g(\varepsilon_1, \varepsilon_2) \subseteq B_g(\varepsilon).$$

Proof. On one hand, $\varepsilon_1 < \varepsilon$, and so, it is easy to check that any numerical semigroup S in $\mathcal{MF}_g(\varepsilon_1, \varepsilon_2)$ satisfies $(\gamma - \varepsilon) < m(\Lambda) < (\gamma + \varepsilon)$. On the other hand,

$$\begin{aligned} F(\Lambda) &> (2-\varepsilon_2)(\gamma-\varepsilon_1)g \\ &= \left(2-\frac{\varepsilon}{2}\right)\left(\gamma - \frac{\gamma\varepsilon}{4+\varepsilon}\right)g \\ &= 2\gamma g - \frac{\gamma\varepsilon g}{2} - \frac{2\gamma\varepsilon g}{4+\varepsilon} + \frac{\gamma\varepsilon^2 g}{2(4+\varepsilon)} \\ &= 2\gamma g - \frac{\gamma\varepsilon g}{2} - \frac{4\gamma\varepsilon g - \gamma\varepsilon^2 g}{2(4+\varepsilon)} \\ &= 2\gamma g - \gamma\varepsilon g + \left(\frac{\gamma\varepsilon g}{2} - \frac{\gamma\varepsilon g(4-\varepsilon)}{2(4+\varepsilon)}\right) \\ &> (2-\varepsilon)\gamma g. \end{aligned}$$

while

$$\begin{aligned} F(\Lambda) &< (2+\varepsilon_2)(\gamma+\varepsilon_1)g \\ &= \left(2+\frac{\varepsilon}{2}\right)\left(\gamma+\frac{\gamma\varepsilon}{4+\varepsilon}\right)g \\ &= 2\gamma g+\frac{\gamma\varepsilon g}{2}+\frac{2\gamma\varepsilon g}{4+\varepsilon}+\frac{\gamma\varepsilon^2 g}{2(4+\varepsilon)} \\ &= 2\gamma g+\frac{\gamma\varepsilon g}{2}+\frac{4\gamma\varepsilon g+\gamma\varepsilon^2 g}{2(4+\varepsilon)} \\ &= 2\gamma g+\frac{\gamma\varepsilon g}{2}+\frac{\gamma\varepsilon g(4+\varepsilon)}{2(4+\varepsilon)} \\ &= (2+\varepsilon)\gamma g. \end{aligned}$$

□

Lemma 4.3. *If $g \geqslant 6$ and $0 < \varepsilon \leqslant \frac{3\gamma-2-\frac{1}{g}}{1+3\gamma}$, then any semigroup in $B_g(\varepsilon)$ has two consecutive left elements.*

Proof. First of all, note that for $g \geqslant 6$ the expression $\frac{3\gamma-2-\frac{1}{g}}{1+3\gamma}$ is positive, so there is some value ε exists in the interval of the hypothesis.

Let $\Lambda \in B_g(\varepsilon)$. Let $\mathcal{L}^*(\Lambda) = \mathcal{L}(\Lambda) \setminus \{0\}$. It holds

$$\begin{aligned} \#\mathcal{L}^*(\Lambda) &= F(\Lambda) - g \\ &> ((2-\varepsilon)\gamma - 1)g \\ &= (2\gamma - 1 - \varepsilon\gamma)g. \end{aligned}$$

If Λ has no two consecutive left elements, then

$$\begin{aligned} F(\Lambda) &\geqslant m(\Lambda) + 2\#\mathcal{L}^*(\Lambda) - 1 \\ &> (\gamma-\varepsilon)g + 2(2\gamma - 1 - \varepsilon\gamma)g - 1 \\ &= (5\gamma - \varepsilon - 2 - 2\varepsilon\gamma)g - 1 \\ &= ((2+\varepsilon)\gamma + 3\gamma - 2 - \varepsilon(1+3\gamma))g - 1 \\ &\geqslant ((2+\varepsilon)\gamma + 3\gamma - 2 - (3\gamma - 2 - \frac{1}{g}))g - 1 \\ &= (2+\varepsilon)\gamma g. \end{aligned}$$

This contradicts the upper bound on the Frobenius number of the elements in $A_g(\varepsilon)$. Hence, Λ must have two consecutive left elements. □

Lemma 4.4. *If $g \geqslant 6$ and $0 < \varepsilon \leqslant \frac{3\gamma - 2 - \frac{1}{g}}{1+3\gamma}$, then $B_g(\varepsilon) \subseteq \overline{S_g^\infty}$.*

Proof. It is a direct consequence of Lemma 4.3 and Theorem 3.5. □

Theorem 4.5. $\lim_{g\to\infty} \frac{\#\overline{S_g^\infty}}{n_g} = 1$.

Proof. Fix ε with $0 < \varepsilon \leqslant \frac{3\gamma - 2 - \frac{1}{g}}{1+3\gamma}$. Set $\varepsilon_1 = \frac{\gamma\varepsilon}{4+\varepsilon}$ and $\varepsilon_2 = \frac{\varepsilon}{2}$. As a consequence of Lemma 4.2 and Lemma 4.4,

$$\mathcal{MF}_g(\varepsilon_1, \varepsilon_2) \subseteq B_g(\varepsilon) \subseteq \overline{S_g^\infty},$$

hence,

$$\lim_{g\to\infty} \frac{\#\overline{S_g^\infty}}{n_g} \geqslant \lim_{g\to\infty} \frac{\#\mathcal{MF}_g(\varepsilon_1, \varepsilon_2)}{n_g}.$$

Now, by Lemma 4.1,

$$\lim_{g\to\infty} \frac{\#\overline{S_g^\infty}}{n_g} \geqslant \lim_{g\to\infty} \frac{\#\mathcal{MF}_g(\varepsilon_1, \varepsilon_2)}{n_g} = 1,$$

which concludes the proof. □

Chapter 5

Generalized Numerical Semigroups: more roots

A ordem das árvores não altera o passarinho.

Tulipa Ruiz, Brazilian singer and songwriter, in the song A Ordem das Árvores

Generalized numerical semigroups (GNS) were introduced in 2016 by Failla, Peterson and Utano [35] and later systematically developed in Cisto's dissertation [25]. As Cisto points out in his work [25], the primary aim of studying GNS is to explore which concepts and results from numerical semigroups can be generalized.

In the course of this chapter, we present basic concepts about GNSs that have been generalized or adapted from numerical semigroups. In general, we will use lowercase and bold letters of the Roman alphabet to represent elements of $\mathbb{N}_0^d,\ d > 1$. We will also denote by $\mathbf{e}_1, \ldots, \mathbf{e}_d$ the standard basis in $\mathbb{R}^d$ and the

d-tuples $(0,\dots,0)$ and $(1,\dots,1)$ will be simply denoted by $\mathbf{0}$ and $\mathbf{1}$, respectively.

The discussion of which order to consider in $\mathbb{N}_0^d$ plays an important role when working with GNSs. Therefore, it is important to look back on the natural partial order $\leqslant$ in $\mathbb{N}_0^d$. For $\mathbf{a} = (\alpha_1,\dots,\alpha_d), \mathbf{b} = (\beta_1,\dots,\beta_d) \in \mathbb{N}_0^d$, we have $\mathbf{a} \leqslant \mathbf{b}$ in $\mathbb{N}_0^d$ if and only if $\alpha_i \leqslant \beta_i$ for all $1 \leqslant i \leqslant d$. Finally, to close this introduction, we define a set that will be useful in what follows. For $\mathbf{a} \in \mathbb{N}_0^d$, we consider the set $C(\mathbf{a}) := \{\mathbf{b} \in \mathbb{N}_0^d : \mathbf{b} \leqslant \mathbf{a}\}$.

5.1 Generalizing the roots

In this section, we explore the generalization of fundamental concepts of numerical semigroups to the setting of generalized numerical semigroups.

Definition 5.1. A *generalized numerical semigroup* (GNS) is a submonoid $S \subseteq \mathbb{N}_0^d$, where d is a positive integer, such that its complement $H(S) = \mathbb{N}_0^d \setminus S$ is finite. As for numerical semigroups, the elements in $H(S)$ are called *gaps* of S and the number $g(S) = \#H(S)$ is called the *genus* of S.

The concept of the conductor in numerical semigroups was generalized to GNS in 2022 by Bernardini, Tenório and Tizziotti [7]. This generalization introduced the notion of the corner of a GNS, which will ensure the finiteness of the set of GNS of a fixed genus g.

Definition 5.2. Let $S \subseteq \mathbb{N}_0^d$ be a GNS. An element $\mathbf{c} = (c_1,\dots,c_d) \in S$ is called a *corner* of S if the following conditions are satisfied:

a. for all $1 \leqslant i \leqslant d$ and for all $\alpha \in \mathbb{N}_0$ such that $\alpha \geqslant c_i$ we have that $(\beta_1,\dots,\beta_{i-1},\alpha,\beta_{i+1},\dots,\beta_d) \in S$ for all $\beta_i,\dots,\beta_{i-1},\beta_{i+1},\dots,\beta_d \in \mathbb{N}_0$;

b. for all $1 \leqslant i \leqslant d$, there exist $\gamma_1,\dots,\gamma_{i-1},\gamma_{i+1},\dots,\gamma_d \in \mathbb{N}_0$ such that $(\gamma_1,\dots,\gamma_{i-1},c_i-1,\gamma_{i+1},\dots,\gamma_d) \notin S$.

It is important to note that every GNS $S \subseteq \mathbb{N}_0^d$ has at least one corner element, as $H(S)$ is finite. Moreover, if $g(S) > 0$, then $H(S) \subseteq C(\mathbf{c} - \mathbf{1})$.

Proposition 5.3. *[7, Proposition 3.2] Let $S \subseteq \mathbb{N}_0^d$ be a GNS. Then the corner of S is unique.*

Proposition 5.4. *[7, Proposition 4.8 and Remark 4.11] Let $S \subseteq \mathbb{N}_0^d$ be a GNS with corner $\mathbf{c} = (c_1, \ldots, c_d)$. If $c_i \geqslant 2$ for all i, then $\sum_{i=1}^{d} c_i \leqslant 2g(S)$. Otherwise, it holds $\sum_{i=1}^{d} c_i \leqslant 2g(S) + d - 1$.*

Given that for a generalized numerical semigroup $S \subseteq \mathbb{N}_0^d$ with corner $\mathbf{c}$, we have $H(S) \subseteq C(\mathbf{c}-\mathbf{1}) \subseteq C(\mathbf{c})$, and since the coordinates of the corner are bounded by $2g(S)+d-1$, we can already conclude that the set of GNSs of genus g is finite. In fact, the number of GNSs in $\mathbb{N}_g^d$ of genus g, denoted by $S_{g,d}$, is bounded by the number of ways to select g gaps within the space $(0, 2g + d - 1]^d \cap \mathbb{N}_0^d$. That is,

$$\#S_{g,d} \leqslant \binom{(2g+d)^d - 1}{g}.$$

We denote the number of elements of $S_{g,d}$ by $n_{g,d}$, that is, $n_{g,d} = \#S_{g,d}$, where $d > 1$ and $g \geqslant 0$.

Example 5.5. There exists a unique GNS of genus 0, namely $\mathbb{N}_0^d$, while for genus 1 there are exactly d GNSs, namely $\mathbb{N}_0^d \setminus \{\mathbf{e}_i\}$ for $1 \leqslant i \leqslant d$. Table 5.1, from [35, Table 1], presents the set of gaps for every GNS of in $\mathbb{N}_0^2$ up to genus 3.

In $\mathbb{N}^d$ there is no natural total order, making it untrivial to define concepts such as enumeration, multiplicity, and Frobenius number. In [35], Failla, Peterson, and Utano introduce the notion of relaxed monomial order, a particular case of a total order, which enables the organization of generalized numerical semigroups into tree structures, similar to those developed for $\mathbb{N}_0$.

Table 5.1: Gaps of GNSs of $\mathbb{N}_0^d$ up to genus 3.

Genus	Gaps
0	$\emptyset$
1	$(1,0)$
	$(0,1)$
2	$(1,0),(1,1)$
	$(1,0),(2,0)$
	$(1,0),(3,0)$
	$(0,1),(0,2)$
	$(0,1),(0,3)$
	$(0,1),(1,0)$
	$(0,1),(1,1)$
3	$(1,0),(1,1),(1,2)$
	$(1,0),(1,1),(2,0)$
	$(1,0),(1,1),(3,0)$
	$(1,0),(2,0),(2,1)$
	$(1,0),(2,0),(3,0)$
	$(1,0),(2,0),(4,0)$
	$(1,0),(2,0),(5,0)$
	$(1,0),(3,0),(5,0)$
	$(0,1),(0,2),(0,3)$
	$(0,1),(0,2),(0,4)$
	$(0,1),(0,2),(0,5)$
	$(0,1),(0,2),(1,0)$
	$(0,1),(0,2),(1,1)$
	$(0,1),(0,2),(1,2)$
	$(0,1),(0,3),(0,5)$
	$(0,1),(0,3),(1,0)$
	$(0,1),(0,3),(1,1)$
	$(0,1),(1,0),(1,1)$
	$(0,1),(1,0),(1,2)$
	$(0,1),(1,0),(2,0)$
	$(0,1),(1,0),(2,1)$
	$(0,1),(1,0),(3,0)$
	$(0,1),(1,1),(2,1)$

The well-known monomial orders are total orders on the set of all (monic) monomials in a given polynomial ring, satisfying the property of compatibility with the ring operation. That is, if $u \leqslant v$ for two monomials u and v, and w is any other monomial, then $u \cdot w \leqslant v \cdot w$. Monomial orders induce total orders in $\mathbb{N}_0^d$.

A monomial order on $\mathbb{N}_0^d$ can be conveniently described in terms of dot products with an ordered set of d linearly independent weight vectors in $\mathbb{R}^d_{\geqslant 0}$ [29, 25]. More precisely, given linearly independent vectors $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_d$ in $\mathbb{R}^d_{\geqslant 0}$, we can define a monomial order $<_m$ on $\mathbb{N}_0^d$ as follows:

$$\mathbf{u} <_m \mathbf{v} \Leftrightarrow \text{ the smallest } i \text{ for which } \mathbf{w}_i \cdot \mathbf{u} \neq \mathbf{w}_i \cdot \mathbf{v} \text{ has } \mathbf{w}_i \cdot \mathbf{u} < \mathbf{w}_i \cdot \mathbf{v}.$$

Definition 5.6. A total order $\prec$ on the elements of $\mathbb{N}_0^d$ is called a *relaxed monomial order* if it satisfies:

a. If $\mathbf{v}, \mathbf{w} \in \mathbb{N}_0^d$ and if $\mathbf{v} \prec \mathbf{w}$ then $\mathbf{v} \prec \mathbf{w} + \mathbf{u}$ for any $\mathbf{u} \in \mathbb{N}_0^d$.

b. If $\mathbf{v} \in \mathbb{N}_0^d$ and if $\mathbf{v} \neq \mathbf{0}$ then $\mathbf{0} \prec \mathbf{v}$.

Example 5.7. Every monomial order is a relaxed monomial order, but the converse does not hold. The *lexicographic order* is a relaxed monomial order in $\mathbb{N}_0^d$ established by selecting the weight vectors $\mathbf{w}_i$ as the canonical ones $\mathbf{e}_i$. In practical terms, given two vectors $\mathbf{u}$ and $\mathbf{v} \in \mathbb{N}_0^d$, we say that $\mathbf{u} \prec \mathbf{v}$ if the first coordinate (looking from left to right) where they differ is smaller in $\mathbf{u}$ than in $\mathbf{v}$. The name of this order is inspired by the way we organize words in the dictionary.

Every permutation of the elements $\mathbf{e}_i$ for the configuration of the weights $\mathbf{w}_i$ is also a relaxed monomial order. This implies a change in the order in which we look for the "first" different coordinate between the vectors we are comparing. In $\mathbb{N}_0^d$, the *colexicographic order* is determined setting $\mathbf{w}_1 = \mathbf{e}_d, \mathbf{w}_2 = \mathbf{e}_{d-1}, \ldots, \mathbf{w}_d = \mathbf{e}_1$.

For $\mathbf{u} = (u_1, \ldots, u_d)$ and $\mathbf{v} = (v_1, \ldots, v_d)$ we can also define the relaxed mono-

mial order $\prec_1$ as $\mathbf{u} \prec_1 \mathbf{v}$ if either $\sum_{i=1}^{d} u_i < \sum_{i=1}^{d} v_i$ or $\sum_{i=1}^{d} u_i = \sum_{i=1}^{d} v_i$ and $\mathbf{u} \prec \mathbf{v}$, where $\prec$ is the lexicographic order. Define the relaxed monomial order $\prec_2$ to be analogous to $\prec_1$ but using the colexicographic order instead of the lexicographic one.

As a final example, we describe a relaxed monomial order that does not satisfy the conditions of a monomial order. Consider again $\prec$ as the lexicographic order on $\mathbb{N}_0^4$, and let $\min \mathbf{u}$ denote the smallest coordinate of $\mathbf{u} = (u_1, u_2, u_3, u_4)$. We define the order $\prec'$ on $\mathbb{N}_0^4$ as follows:

$$\mathbf{u} \prec' \mathbf{v} \text{ if either}$$

(i) $\min \mathbf{u} \prec \min \mathbf{v}$ or

(ii) $\min \mathbf{u} = \min \mathbf{v}$ and $\mathbf{u} \prec \mathbf{v}$

It is straightforward to verify that this constitutes a relaxed monomial order. However, it fails to be a monomial order. For instance, consider the elements $\mathbf{u} = (1, 2, 3, 4), \mathbf{v} = (1, 2, 4, 3)$, and $\mathbf{w} = (4, 3, 1, 0)$. Then, despite having $\mathbf{u} \prec' \mathbf{v}$, we obtain $\mathbf{v} + \mathbf{w} \prec' \mathbf{u} + \mathbf{w}$, violating the multiplicative property required for monomial orders.

Once a total order is established on $\mathbb{N}_0^d$, the Frobenius number of a GNS can be defined analogously to the classical case of numerical semigroups. Moreover, since a relaxed monomial order on $\mathbb{N}_0^d$ is a well-ordering, it is also possible to define the multiplicity of a GNS.

Definition 5.8. Let $S \subseteq \mathbb{N}_0^d$ be a GNS and $\prec$ be a relaxed monomial order in $\mathbb{N}_0^d$. The *multiplicity of S with respect to $\prec$*, $m_\prec(S)$, is the smallest non-zero element of S. The *Frobenius element of S with respect to $\prec$*, $F_\prec(S)$, is the largest element of $H(S)$ with respect to $\prec$. For convenience, we denote $F_\prec(\mathbb{N}_0^d)$ as $(-1, \ldots, -1)$.

The *left elements of S with respect to $\prec$*, $\mathcal{L}_{\prec}(S)$, is the set of elements of S that are smaller than $F_{\prec}(S)$ with respect to $\prec$.

As introduced in [27], the notion of an ordinary generalized numerical semigroup is defined as follows. It provides a natural generalization of the concept of an ordinary numerical semigroup to higher dimensions.

Definition 5.9. A GNS $S \subseteq \mathbb{N}_0^d$ is called *ordinary* if there exists some $\mathbf{s} \in \mathbb{N}_0^d$ such that $S = \{\mathbf{0}\} \cup (\mathbb{N}_d^0 \setminus C(\mathbf{s}))$.

Lemma 5.10. *[7, Lemma 4.3] Let $S = \{\boldsymbol{0}\} \cup (\mathbb{N}_0^d \setminus C(\mathbf{s})) \subseteq \mathbb{N}_0^d$ be an ordinary GNS. Then the corner of S is $\mathbf{c} = \mathbf{s} + \boldsymbol{1}$.*

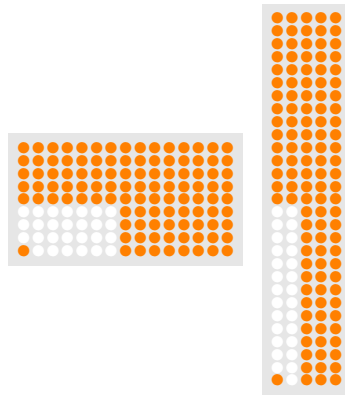
Following the notation introduced in [7], we denote the ordinary GNS with corner $\mathbf{c}$ by $\mathcal{O}(\mathbf{c})$. Recall that the notation $\mathcal{O}_g$ represents the numerical semigroup of genus g, which has multiplicity and conductor equal to $g+1$. In the context of GNS, $\mathcal{O}(\mathbf{c})$ represents the GNS with corner $\mathbf{c} = (c_1, \ldots, c_d)$, genus $\prod_{i=1}^{d} c_i - 1$, and a multiplicity that depends on the chosen relaxed monomial order. This implies that, for a fixed genus g, there may be multiple ordinary GNSs with genus g. We now present examples of $\mathcal{O}(\mathbf{c})$ in which the multiplicity differs from the corner.

Example 5.11. In Figure 5.1 we can see ordinary GNSs in $\mathbb{N}_0^2$ with corner $(7,4)$ and $(2,14)$, and multiplicity $(0,4)$ and $(2,0)$ with respect to $\prec_1$ (order presented in Example 5.7). Note that both $\mathcal{O}((7,4))$ and $\mathcal{O}((2,14))$ are ordinary GNSs of genus 27.

Analogously to numerical semigroups, a set $A \subseteq \mathbb{N}_0^d$ is said to be a generating set of a generalized numerical semigroup S if every element of S can be expressed as a sum of elements in A, that is, $S = \{\lambda_1\mathbf{a}_1 + \cdots + \lambda_n\mathbf{a}_n : \lambda_1, \ldots, \lambda_n \in \mathbb{N}_0, \mathbf{a}_1, \ldots, \mathbf{a}_n \in A\}$. In this case, we write $S = \langle A \rangle$.

Proposition 5.12. *[28, Corollary 2.4] Every GNS admits a unique finite system of minimal generators.*

Figure 5.1: On the left, $\mathcal{O}((7,4)) \subset \mathbb{N}_0^2$; on the right, $\mathcal{O}((2,14)) \subset \mathbb{N}_0^2$.



The following theorem, the main result of Cisto, Failla and Utano in [28], establishes the conditions under which a set $A \subseteq \mathbb{N}_0^d$ generates a generalized numerical semigroup. We will state this in a different notation from that of the original authors. To this end, we define

$$\begin{aligned} E_k(A) &= \{a \in \mathbb{N}_0 : a\mathbf{e}_k \in A\}, \\ U_{i,j}(A) &= \{a \in \mathbb{N}_0 : a\mathbf{e}_i + e_j \in A\}. \end{aligned}$$

Theorem 5.13. *[28, Theorem 2.8] Let $d \geqslant 2$. Given a set $A \subseteq \mathbb{N}^d$, $\langle A \rangle$ is a GNS if and only if the two following statements hold:*

a. For every k with $1 \leqslant k \leqslant d$, there is a finite subset of coprime elements in $E_k(A)$;

b. For every i, j with $1 \leqslant i < j \leqslant d$, $U_{i,j}(A) \neq \emptyset$.

Definition 5.14. Let $\prec$ be a relaxed monomial order on $\mathbb{N}_0^d$ and $S \subseteq \mathbb{N}_0^d$ be a generalized numerical semigroup. An element $\mathbf{s} \in S$ is an *effective generator of S with respect to $\prec$* if $\mathbf{s}$ is a minimal generator of S and $F_\prec(S) \prec \mathbf{s}$. We will denote

by $U_{\prec}(S)$ the set of effective generators of S with respect to $\prec$.

Proposition 5.15. *[35, Proposition 4.1] Let $S \subseteq \mathbb{N}_0^d$ be a generalized numerical semigroup, $\prec$ be a relaxed monomial order in $\mathbb{N}_0^d$ and $\mathbf{h}$ a minimal generator of S. The set $T = S \cup \{F_{\prec}(S)\}$ is also a generalized numerical semigroup, and moreover, $F_{\prec}(S)$ is an effective generator of T with respect to $\prec$. Likewise, the set $T = S \setminus \{\mathbf{h}\}$ is a generalized numerical semigroup. Furthermore, if $\mathbf{h}$ is an effective generator of S, then $F_{\prec}(T) = \mathbf{h}$.*

To conclude this section on the foundations of generalized numerical semigroups, we present the following theorem, demonstrated in [35]. It reveals aspects on the asymptotic behavior of $n_{g,d}$.

Theorem 5.16. *[35, Theorem 5.3] For fixed genus g, $n_{g,d}$ is a degree g polynomial in $\mathbb{Q}[d]$.*

Example 5.17. [35, Example 5.4]

$$n_{1,d} = d, \qquad n_{2,d} = \frac{3}{2}d^2 + \frac{1}{2}d, \qquad n_{3,d} = \frac{5}{3}d^3 + \frac{5}{2}d^2 - \frac{1}{6}d.$$

5.2 Trees of generalized numerical semigroups

As in the case of numerical semigroups, Proposition 5.15 suggests that every generalized numerical semigroup of genus $g+1$ is obtained from a generalized numerical semigroup S of genus g by removing one of its effective generators. In this way, given a relaxed monomial order $\prec$ in $\mathbb{N}_0^d$, we can define the tree $\mathcal{T}_{\prec}^d$, whose root is $\mathbb{N}_0^d$ and whose nodes at a given depth correspond to all GNSs of genus equal to that depth. In this section, we will provide some examples of trees of GNSs that we can build.

Example 5.18. We will use this type of configuration when drawing a GNS: in white the gaps and in red the generators, as illustrated with $\mathcal{O}((2,2))$, by order $\prec_1$, in Figure 5.2.

Figure 5.2: Ordinary generalized numerical semigroup of corner $(2, 2)$, represented by order $\prec_1$.



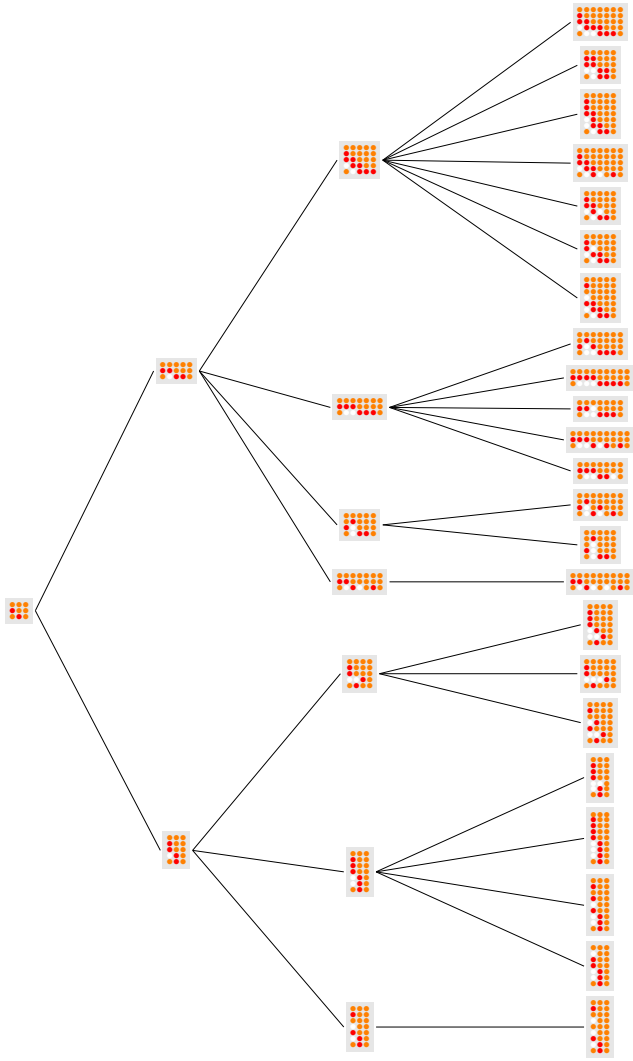
The parent of a GNS S in $\mathcal{T}^d_\prec$ is given by $S \cup F_\prec(S)$. Conversely, the children of a GNS S can be obtained by taking away one by one all the minimal generators of S that are larger by $\prec$ than $F_\prec(S)$. Thus, we emphasize the plural form of the word "tree" in the title of this section, as, although the construction method remains the same, different trees of GNSs organized by genus can be obtained depending on the chosen relaxed monomial order. The important thing is that, even if the kinship relations between GNSs of consecutive levels in each possible tree change, the number of GNSs at each level g of a tree $\mathcal{T}^d_\prec$ is the same for any given $\prec$. This occurs as every generalized numerical semigroup of genus g is generated by eliminating each effective generator from those GNSs of genus $g-1$. Cisto, Delgado and García-Sánchez, in [26], consequently proposed the following lemma.

Lemma 5.19. *[26, Corollary 13] Let $g, d \in \mathbb{N}_0$ and $\prec$ be a relaxed monomial order. Then*

$$n_{g,d} = \sum_{S \in S_{g-1,d}} \#U_\prec(S).$$

Example 5.20. Consider $\prec_1$ in $\mathbb{N}_0^2$ as in Example 5.7. The tree $\mathcal{T}^2_{\prec_1}$ up to genus 3 is illustrated in Figure 5.3.

Figure 5.3: Tree $\mathcal{T}^2_{\prec_1}$ of GNSs up to genus 3.



Chapter 6

Infinite Chains of Generalized Numerical Semigroups

> Se você tem uma ideia incrível, é melhor fazer uma canção.
>
> Caetano Veloso, Brazilian singer and musician, in the song Língua

Let $\prec$ be a relaxed monomial order in $\mathbb{N}_0^d$ with $d > 1$. Analogously to the definition of infinite chains in the context of numerical semigroups, an infinite chain in the tree $\mathcal{T}_{\prec}^d$ is a sequence of generalized numerical semigroups $\{S_i\}_{i\in\mathbb{N}_0}$ such that $S_0 = \mathbb{N}_0^d$ and each S_i is the parent of S_{i+1} in $\mathcal{T}_{\prec}^d$. Whenever the order considered in $\mathcal{T}_{\prec}^d$ is clear from the context, the notion of descendants of a node in the tree remains the same as in the case of numerical semigroups. In this way, a generalized numerical semigroup belongs to an infinite chain in $\mathcal{T}_{\prec}^d$ if and only if it has infinitely many descendants in this tree.

In this chapter, we characterize the conditions under which a node in $\mathcal{T}_{\prec}^d$

belongs to an infinite chain. Additionally, we explore, through examples, the behavior of these chains when the order in $\mathbb{N}_0^d$ is changed. In the context of having multiple GNS trees, we will see that for distinct relaxed monomial orders $\prec_1$ and $\prec_2$, a given GNS S may belong to an infinite chain in $\mathcal{T}^d_{\prec_1}$ while having only a finite number of descendants in $\mathcal{T}^d_{\prec_2}$.

The consideration of infinite chains in $\mathcal{T}^d_{\prec}$ is warranted for every $\prec$. More precisely, one should examine the generalized numerical semigroup $\mathcal{H}^i_g \subseteq \mathbb{N}_0^d$, $i > 0$, defined by

$$\mathcal{H}^i_g := \mathbb{N}_0^d \setminus (\{0\} \times \ldots \times \{0\} \times \underbrace{H(\mathcal{H}_g)}_{i-\text{th position}} \times \{0\} \times \ldots \times \{0\}).$$

It can be readily verified that $\mathcal{H}^i_g$ constitutes a GNS with g gaps. Moreover, for any relaxed monomial order $\prec$ established on $\mathbb{N}_0^d$, one has

$$(0, \ldots, 0, \underbrace{a}_{i-\text{th position}}, 0, \ldots, 0) \prec (0, \ldots, 0, \underbrace{b}_{i-\text{th position}}, 0, \ldots, 0)$$

if and only if $a < b$. This ensures that $F_{\prec}(\mathcal{H}^i_g) = (0, \ldots, 0, \underbrace{2g-1}_{i-\text{th position}}, 0, \ldots, 0)$ and $(0, \ldots, 0, \underbrace{2g+1}_{i-\text{th position}}, 0, \ldots, 0)$ is the unique effective generator of $\mathcal{H}^i_g$. This means that for every $1 \leqslant i \leqslant d$ the sequence $\{\mathcal{H}^i_g\}_g$ is an infinite chain in $\mathcal{T}^d_{\prec}$.

Recalling item a of Theorem 3.5 of Chapter 3, which characterizes numerical semigroups that belong to infinite chains, we observe that the criterion is that the left elements of the semigroup generate a numerical semigroup. This insight motivated the following theorem, which extends Theorem 3.5.a to higher dimensions.

Theorem 6.1. *A GNS S is not in an infinite chain in $\mathcal{T}^d_{\prec}$ if and only if the set $\mathcal{L}_{\prec}(S)$ satisfies both*

a. For every k with $1 \leqslant k \leqslant d$, the elements in $E_k(\mathcal{L}_\prec(S))$ are coprime;

b. For every i, j with $1 \leqslant i < j \leqslant d$, $U_{i,j}(\mathcal{L}_\prec(S)) \neq \emptyset$.

Proof. If the set $\mathcal{L}_\prec(S)$ generates a GNS, the maximum level at which there are descendants of S is at most the genus of the GNS generated by $\mathcal{L}_\prec(S)$. In fact, any descendant of S must contain $\mathcal{L}_\prec(S)$ and therefore must contain the GNS generated by $\mathcal{L}_\prec(S)$. Hence, the total number of descendants of S is finite and cannot belong to any infinite chain.

On the other hand, if the set $\mathcal{L}_\prec(S)$ does not generate a GNS, then S belongs to the infinite chain defined as follows: Let $a_0, a_1, \dots$ be the enumeration of $\mathbb{N}_0^d$ by order $\prec$. Define A_i as the union

$$A_i = \langle \mathcal{L}_\prec(S) \rangle \cup \{a_i, a_{i+1}, \dots\}$$

If we take away the repeated GNSs in the sequence $(A_i)_i$, then we obtain an infinite chain of GNSs. □

Theorem 6.1 is a generalization of the version for numerical semigroups (Theorem 3.5 item a). In fact, for $d = 1$ we can define $E_1(\mathcal{L}(S)) := \{a \in \mathbb{N}_0 : a\mathbf{e}_1 \in \mathcal{L}(S)\}$, which is equal to $\mathcal{L}(S)$. Since there are no i, j such that $1 \leqslant i < j \leqslant d$, $U_{i,j}(\mathcal{L}(S))$ is not defined.

Example 6.2. Consider in $\mathbb{N}_0^2$ the relaxed monomial order $\prec_1$ presented in Example 5.7. In the tree in Figure 6.1 we colored the edges according to whether they belong to an infinite chain or not. If an edge points to a semigroup S whose

- set $E_0(\mathcal{L}_\prec(S))$ is not coprime, then the edge is red;
- set $E_0(\mathcal{L}_\prec(S))$ is coprime, but whose set $E_1(\mathcal{L}_\prec(S))$ is not, then the edge is green;

- sets $E_0(\mathcal{L}_\prec(S))$ and $E_1(\mathcal{L}_\prec(S))$ are coprime, but $U_{1,2}(\mathcal{L}_\prec(S)) = \emptyset$, then the edge is blue;
- sets $E_0(\mathcal{L}_\prec(S))$ and $E_1(\mathcal{L}_\prec(S))$ are coprime, whose set $U_{1,2}(\mathcal{L}_\prec(S)) \neq \emptyset$, and such that $U_{1,2}(\mathcal{L}_\prec(S)) = \emptyset$, then the edge is yellow.

Otherwise, the edge belongs only to finite chains and it is black.

The first example of a GNS in the tree of Figure 6.1 pointed by a blue edge has genus 4 and is represented in Figure 6.2.

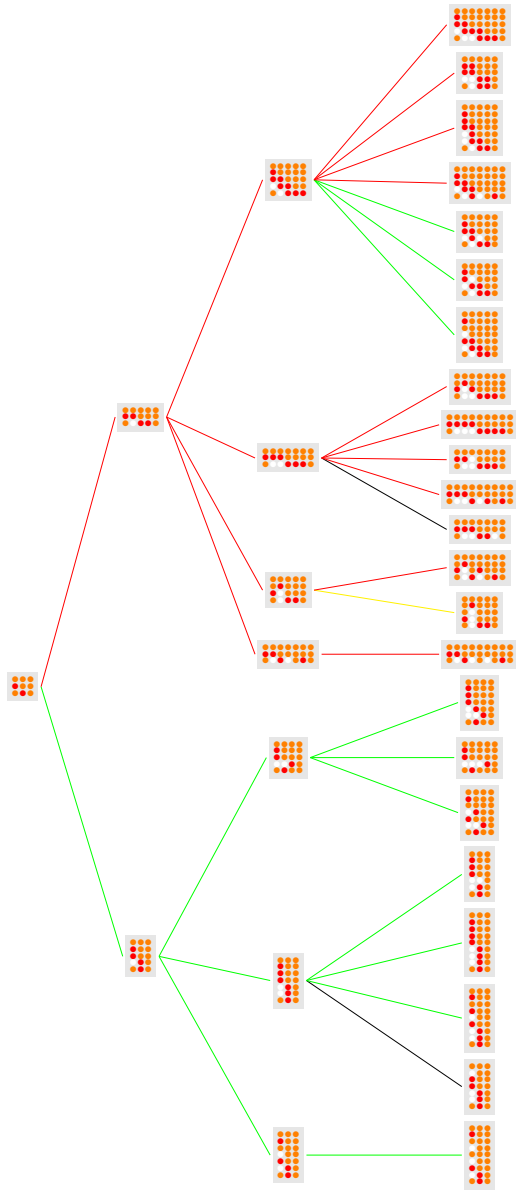
6.1 Changing the order

In this section, we explore how the choice of a relaxed monomial order $\prec$ influences the inclusion of a generalized numerical semigroup S in infinite chains within the associated tree structure, demonstrating that S may not belong to any infinite chain in the tree associated with a given total order $\prec_a$ and at the same time belong to an infinite chain in the tree associated with a different order $\prec_b$. Furthermore, in Theorem 6.3 we establish the conditions under which a generalized numerical semigroup S belongs to infinitely many infinite chains. First, the case of the semigroup $S \subset \mathbb{N}_0^2$ in Figure 6.3 is considered.

The semigroup S in Figure 6.3 does not belong to any infinite chain with respect to the order $\prec_1$ from Example 5.7. But it belongs to an infinite chain with respect to the order $\prec_2$ (also from Example 5.7). Indeed, since $\mathcal{L}_{\prec_1}(S) = \{(0,0),(3,0),(4,0),(2,1),(0,2),(1,2),(0,3)\}$ and $\mathcal{L}_{\prec_2}(S) = \{(0,0),(3,0),(2,1),(0,2),(1,2),(2,2),(0,3),(1,3),(0,4)\}$, the application of Theorem 6.1 permits us to validate the assertion.

Analyzing the conditions that influence whether a given GNS belongs to infinitely many infinite chains provides insight into the hierarchical organization of these objects. The following theorem identifies key properties of the sets

Figure 6.1: Colored structure of tree $\mathcal{T}^2_{\prec_1}$ up to genus 3.



$E_k(\mathcal{L}_\prec(S))$ and $U_{i,j}(\mathcal{L}_\prec(S))$ that determine whether S belongs to no infinite chain or belongs to infinitely many of them.

Figure 6.2: GNS of genus 4 whose edge is pointed by blue in $\mathcal{T}^2_{\prec_1}$.



Figure 6.3: GNS in $\mathbb{N}_0^2$ that belongs to distinct quantities of infinite chains depending on the chosen order.



Theorem 6.3. *Let $S \subseteq \mathbb{N}_0^d$ be a GNS and $\prec$ be a relaxed monomial order in $\mathbb{N}_0^d$. Let $k_1, \ldots, k_r$, $i_1, \ldots, i_s$, $j_1, \ldots, j_s$ be such that $E_{k_1}(\mathcal{L}_\prec(S)), \ldots, E_{k_r}(\mathcal{L}_\prec(S))$ and $U_{i_1,j_1}(\mathcal{L}_\prec(S)), \ldots, U_{i_s,j_s}(\mathcal{L}_\prec(S))$ are the unique sets $E_k(\mathcal{L}_\prec(S))$ and $U_{i,j}(\mathcal{L}_\prec(S))$ such that $\gcd(E_{k_\ell}(\mathcal{L}_\prec(S))) \neq 1$ and $U_{i_\ell,j_\ell}(\mathcal{L}_\prec(S)) = \emptyset$. If*

a. $r + s = 0$, then S does not belong to any infinite chain of $\mathcal{T}^d_\prec$;

b. $r + s \geqslant 2$ then S belongs to infinitely many infinite chains of $\mathcal{T}^d_\prec$.

Proof. If $r + s = 0$, then S does not belong to any infinite chain by Theorem 6.1.

For $\ell \in \{1, \ldots, r\}$, let

$$d_\ell = \gcd(E_{k_\ell}(\mathcal{L}_\prec(S))),$$

$$a_\ell = \min\{a \in \mathbb{N}_0 : (a \cdot d_\ell)\mathbf{e}_{k_\ell} \succ F_\prec(S)\},$$

and, for $\ell \in \{1, \ldots, s\}$, let

$$b_\ell = \min\{b \in \mathbb{N}_0 : b\mathbf{e}_{i_\ell} + \mathbf{e}_{j_\ell} \succ F_\prec(S)\}.$$

Now, for any $(m_1, \ldots, m_r, n_1, \ldots, n_s) \in \mathbb{N}_0^{r+s}$, define

$$S_{m_1,\ldots,m_r,n_1,\ldots,n_s} := \langle \mathcal{L}_{\prec}(S) \cup \{((a_\ell + m_\ell) \cdot d_\ell + 1)\mathbf{e}_{k_\ell}\}_{1 \leqslant \ell \leqslant r} \cup \{(b_\ell + n_\ell)\mathbf{e}_{i_\ell} + \mathbf{e}_{j_\ell}\}_{1 \leqslant \ell \leqslant s} \rangle.$$

One can check that

- $S_{m_1,\ldots,m_r,n_1,\ldots,n_s}$ is a GNS;
- $S \supseteq S_{m_1,\ldots,m_r,n_1,\ldots,n_s}$;
- If $m'_1 \geqslant m_1, \ldots, m'_r \geqslant m_r, n'_1 \geqslant n_1, \ldots, n'_s \geqslant n_s$, then $S_{m_1,\ldots,m_r,n_1,\ldots,n_s} \supseteq S_{m'_1,\ldots,m'_r,n'_1,\ldots,n'_s}$;
- There is a unique chain in $\mathcal{T}^d_{\prec}$ from S to $S_{0,\ldots,0}$;

  In fact, if S is a GNS with gaps $h_1 \prec h_2 \prec \ldots \prec h_g$ and S' is a GNS with gaps $h_1 \prec h_2 \prec \ldots \prec h_g \prec h_{g+1} \prec \ldots \prec h_{g'}$, then the unique chain from S to S' is

  $$S \subset S \setminus \{h_{g+1}\} \subset S \setminus \{h_{g+1}, h_{g+2}\} \subset S \setminus \{h_{g+1}, h_{g+2}, h_{g+3}\} \subset \ldots;$$

- For any $(m_1, \ldots, m_r, n_1, \ldots, n_s), (m'_1, \ldots, m'_r, n'_1, \ldots, n'_s) \in \mathbb{N}_0^{r+s}$, if $m'_1 \geqslant m_1, \ldots, m'_r \geqslant m_r, n'_1 \geqslant n_1, \ldots, n'_s \geqslant n_s$, there is a unique chain in $\mathcal{T}^d_{\prec}$ from $S_{m_1,\ldots,m_r,n_1,\ldots,n_s}$ to $S_{m'_1,\ldots,m'_r,n'_1,\ldots,n'_s}$.

Now, suppose $r + s \geqslant 2$. If we fix $m_1, \ldots, m_r, n_1, \ldots, n_{s-1}$ (or $m_1, \ldots, m_r$ in case $s = 1$ or $m_1, \ldots, m_{r-1}$ in case $s = 0$), then the set of GNSs

$$(S_{m_1,\ldots,m_r,n_1,\ldots,n_s})_{n_s \in \mathbb{N}_0}$$

(or $(S_{m_1,\ldots,m_r,n_1})_{n_1 \in \mathbb{N}_0}$ in case $s = 1$ or $(S_{m_1,\ldots,m_r})_{m_r \in \mathbb{N}_0}$ in case $s = 0$) together with $\mathbb{N}^d$ and the intermediate paths conforms an infinite chain containing S. Now, by varying $m_1, \ldots, m_r, n_1, \ldots, n_{s-1} \in \mathbb{N}^{r+s-1}$ (or $m_1, \ldots, m_r$ in case $s = 1$ or

$m_1, \ldots, m_{r-1}$ in case $s = 0$), we obtain an infinite number of infinite chains containing S. □

As we shall see next, there are instances of GNSs for which $r + s = 1$ belong to infinitely many infinite chains, and other instances that are part of just a limited number of infinite chains.

Example 6.4. The semigroup in Figure 6.4 has $r = 1$ and $s = 0$, and belongs to a unique infinite chain in $\mathcal{T}^2_{\prec_1}$, namely that one with one piece shown in Figure 6.5. Although the semigroup in Figure 6.6 also has $r = 1$ and $s = 0$, it belongs to infinitely many infinite chains, some of them are presented in Figure 6.7.

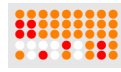
Figure 6.4: GNS in $\mathbb{N}_0^2$ that belongs to a single infinite chain in $\mathcal{T}^2_{\prec_1}$.



Figure 6.5: A part of the infinite chain to which the GNS in Figure 6.4 belongs.



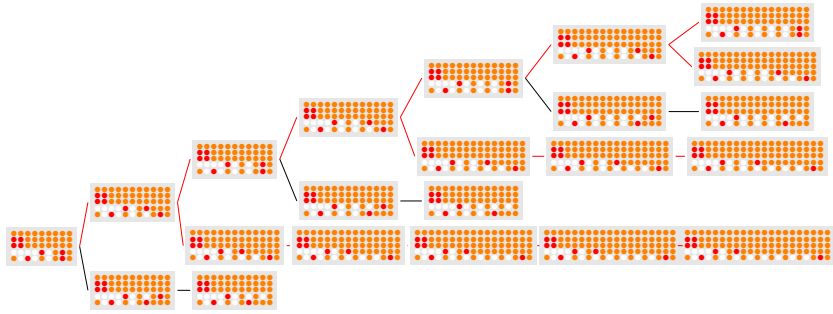
Figure 6.6: GNS in $\mathbb{N}_0^2$ that belongs infinitely many infinite chains in $\mathcal{T}^2_{\prec_1}$.



6.2 Changing the dimension

We conclude this chapter by presenting several additional computations that illustrate the behavior of infinite chains at the initial levels in one-, two-, and three-dimensional spaces. In Figure 6.8 one can see the behavior of infinite chains up to genus 8 for dimension 1. In Figure 6.9 one can see the behavior of infinite chains up to genus 6 for dimension 2 and the same order $\prec_1$ as before.

Figure 6.7: Some infinite chains to which the GNS in Figure 6.6 belongs to.



In Figure 6.10 one can see the behavior of infinite chains up to genus 4 for dimension 3 and also with the order $\prec_1$. Now we color the edges according to whether they belong to an infinite chain or not as follows. If an edge points to a semigroup S whose

- set $E_1(\mathcal{L}_{\prec_1}(S))$ is not coprime, the edge is red;
- set $E_1(\mathcal{L}_{\prec_1}(S))$ is coprime, but whose set $E_2(\mathcal{L}_{\prec_1}(S))$ is not, then the edge is green;
- sets $E_1(\mathcal{L}_{\prec_1}(S))$ and $E_2(\mathcal{L}_{\prec_1}(S))$ are coprime, but whose set $E_3(\mathcal{L}_{\prec_1}(S))$ is not, then the edge is orange;
- sets $E_1(\mathcal{L}_{\prec_1}(S))$, $E_2(\mathcal{L}_{\prec_1}(S))$, and $E_3(\mathcal{L}_{\prec_1}(S))$ are coprime, but $U_{1,2}(\mathcal{L}_{\prec_1}(S)) = \emptyset$ or $U_{1,3}(\mathcal{L}_{\prec_1}(S)) = \emptyset$, then the edge is blue;
- sets $E_1(\mathcal{L}_{\prec_1}(S))$, $E_2(\mathcal{L}_{\prec_1}(S))$, and $E_3(\mathcal{L}_{\prec_1}(S))$ are coprime, whose sets $U_{1,2}(\mathcal{L}_{\prec_1}(S))$ and $U_{1,3}(\mathcal{L}_{\prec_1}(S))$ are not empty, and such that $U_{2,1}(\mathcal{L}_{\prec_1}(S)) = \emptyset$ or $U_{2,3}(\mathcal{L}_{\prec_1}(S)) = \emptyset$, then the edge is yellow;
- sets $E_1(\mathcal{L}_{\prec_1}(S))$, $E_2(\mathcal{L}_{\prec_1}(S))$, and $E_3(\mathcal{L}_{\prec_1}(S))$ are coprime, whose sets $U_{1,2}(\mathcal{L}_{\prec_1}(S))$ and $U_{1,3}(\mathcal{L}_{\prec_1}(S))$, $U_{2,1}(\mathcal{L}_{\prec_1}(S))$ an $U_{2,3}(\mathcal{L}_{\prec_1}(S))$ are not empty, and such that $U_{3,1}(\mathcal{L}_{\prec_1}(S)) = \emptyset$ or $U_{3,2}(\mathcal{L}_{\prec_1}(S)) = \emptyset$, then the edge is pink.

Otherwise, the edge belongs to a finite chain and it is black.

Figure 6.8: Colored structure of the tree $\mathcal{T}$ up to genus 8.

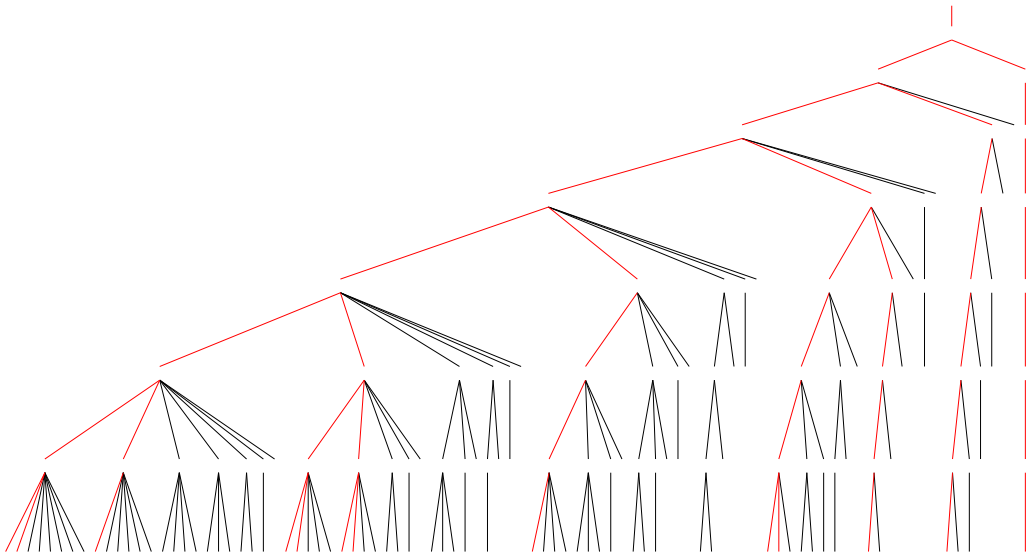


Figure 6.9: Colored structure of the tree $\mathcal{T}^2_{\prec_1}$ up to genus 6.

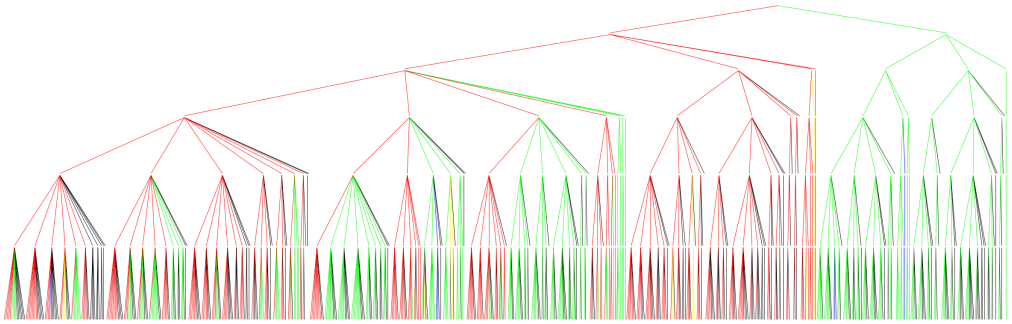
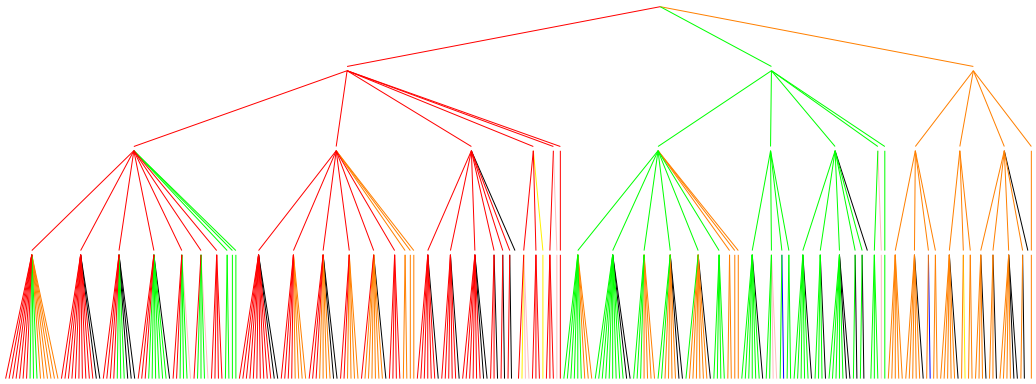


Figure 6.10: Colored structure of tree $\mathcal{T}^3_{\prec_1}$ up to genus 4.



With regard to these three dimensions, as the dimension increases, the proportion of infinite chains increases. However, these computations are at very early levels of the trees. Thus, we expect that, although infinite chains also support GNS trees, they will be rare as the genus increases.

6.3 Open questions

We present below a set of questions that will guide the further development of this research on infinite chains in generalized numerical semigroups.

- Under the conditions of Theorem 6.3, how to characterize when a GNS with $r + s = 1$ belongs to a finite number of infinite chains?
- Are there fewer GNSs in infinite chains than outside them, and does this depend on the ordering or the dimension of the tree?
- Which orderings have the most semigroups in infinite chains? Which orderings have the least?
- What kind of subtree can we identify in generalized numerical semigroup trees, such as the fixed-multiplicity trees in Section 3.2 of Chapter 3? In

particular, how to define a hyperelliptic generalized numerical semigroup?

Chapter 7

Conclusions and Future Work

The study of numerical semigroups has evolved significantly in recent decades, revealing deep connections with combinatorics, algebra, and computational mathematics. This thesis contributes to this growing field by initiating the examination of infinite chains within trees of numerical semigroups and their generalizations. Through this work, we have derived new structural results and developed formulas to count numerical semigroups in infinite chains for specific cases, shedding light on the nature of these objects. In this last chapter, the main findings are summarized and the results are compared with prior research. Open problems and future research directions are also emphasized.

7.1 Summary of key contributions

- Rarity of infinite chains: We proved that infinite chains, while supporting the tree of numerical semigroups, are rare. This resolves an open problem dating back to 2009.
- Explicit counting formulas: We derived explicit formulas for counting numerical semigroups in infinite chains when the multiplicity is prime, 4, or

6. These formulas suggest that the counting function may exhibit quasi-polynomial behavior for any multiplicity.

- Extension to generalized numerical semigroups: We broadened our analysis to include generalized numerical semigroups, characterizing their infinite chains and demonstrating that a given semigroup may belong to an infinite chain under one ordering but not under another. This highlights the complexity of infinite chains in the generalized setting.

7.2 Structural properties of infinite chains identified

- Scarcity of fertile semigroups: At each level g, the number of semigroups belonging to infinite chains is less than half of the total number of semigroups at that level. This phenomenon arises due to the scarcity of, as defined by us, fertile semigroups.

- Patterns in infinite-chain subtrees: For fixed multiplicity m, we identified a repeating pattern in the subtrees of infinite chains. An initial form with $m-1$ levels is replicated throughout the subtree via push by multiplicity. This pattern allowed us to predict the complete behavior of infinite-chain subtrees for multiplicities up to 7.

7.3 Integration with existing literature

- Asymptotic results as a tool: To establish the rarity of infinite chains, we leveraged asymptotic results from [42]. Specifically, we constructed sets that simultaneously satisfy these results and proved that they are contained within the set of numerical semigroups of genus g that belong to finite chains. As a consequence, we obtained that as the genus tends to infinity, the ratio between the number of semigroups in finite chains of genus g and the total number of numerical semigroups of genus g approaches 1, aligning with computational expectations.

- Computations versus rarity: The rarity of infinite chains underscores the peculiar behavior of the numerical semigroup tree when leveled by genus and, consequently, the growth of the sequence n_g. According to the most recent computations by Bras-Amorós [21], the ratio n_{77}/n_{76} spans eleven orders of magnitude. While these calculations point to the rapid growth of n_g — and thus the rapid expansion of the tree itself — the structure that appears to support this growth, namely infinite chains, becomes increasingly rare as we go up the tree.

- Quasi-polynomiality and counting formulas: Our explicit counting formulas for multiplicities $m = 4, 6$, and prime values suggest a quasi-polynomial behavior. This observation aligns with the quasi-polynomial properties studied in [41] and [3], reinforcing the idea that numerical semigroup subtrees exhibit structured patterns.

- Characterization of GNS in finite chains: Building on the foundational work of [35], we characterize generalized numerical semigroups (GNS) that belong to finite chains. Specifically, we use their main result, which establishes necessary and sufficient conditions for a set to generate a GNS. By applying these conditions, we identify the structural properties that ensure a GNS is part of a finite chain, thereby extending the characterization previously established for numerical semigroups in [15].

7.4 Open problems and prospective research directions

- Generalizing counting formulas: A natural next step is to compute $i_g(m)$ for any m, using the pattern offered by push and obtaining quasi-polynomial expressions.

- Infinite chains in generalized numerical semigroups: Further research is needed to determine when a generalized numerical semigroup belongs to

only a finite number of infinite chains and to calculate the number of chains to which it belongs.

7.5 Closing Remarks

Beyond numerical semigroups, our extension to generalized numerical semigroups demonstrates that the study of these objects is far from complete. Mathematics often progresses through the interplay of abstraction and computation. The push operation introduced in this work and its relation with the quasi-polynomial nature of our counting functions provide new tools that may prove useful beyond the specific problems addressed here. These findings not only contribute to a deeper understanding of numerical and generalized numerical semigroups but also suggest potential applications in related areas, such as enumerative combinatorics and computational algebra.

Much like the trees in Thoreau's reflection in the epigraph of this thesis, mathematical ideas often take time to develop strong roots. The slow and deliberate growth of knowledge — building upon existing frameworks, refining concepts, and uncovering hidden patterns — leads to a more profound and resilient grasp. This work represents a step in that ongoing process. While many questions remain open, the results presented here provide a basis upon which future research can build, fostering deeper insights into the intricate and beautiful world of numerical semigroups.

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