Fuzzy Measures and Comonotonicity on Multisets

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Abstract. Fuzzy measures on multisets are studied. We show that a class of multisets can be represented as a subset of positive integers. Comonotonicity for multisets are defined. We show that a fuzzy measure on multisets with some comonotonicity condition can be represented by generalized fuzzy integral.

Keywords: Fuzzy measure, multiset, Choquet integral, Sugeno integral, Generalized fuzzy integral.

1 Introduction

Fuzzy measures [15] are set functions $\mu : \wp(X) \to [0,1]$ that permits us to represent interactions between the elements of the set. Fuzzy measures are often used in conjunction with fuzzy integrals to aggregate information from several sources [17].

Several extensions and variations for fuzzy measures exist. E.g. [4,10] deal with discrete fuzzy measures (i.e., $\mu : \wp(X) \to L$ where L is an ordinal scale), [5] considered set-valued measures.

In this paper we discuss another type of extension. Here, fuzzy measures are defined for multisets. Multisets (or bags) [6,11,19] are a generalization of sets in which multiple appearances of an element is permitted.

The structure of the paper is as follows. In Section 2 we review some previous results needed in this paper. In Section 3 we discuss about the representation of fuzzy measures on multisets. In Section 4 we introduce comonotonicity on multisets and give some results. In Section 5 we have a proposition that a fuzzy measure on multisets can be represented by a generalized fuzzy integral when some comonotonicity conditions hold.

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2 Preliminaries

In this section we review the concepts of fuzzy measures and two fuzzy integrals: the Choquet and Sugeno integrals.

2.1 Choquet Integral and Sugeno Integral

We present here the definition of fuzzy measures and the ones of Choquet and Sugeno integrals. In this paper we will use $\vee$ and $\wedge$ to denote, respectively, the maximum and the minimum.

Definition 1. *Let X be a universal set and $\mathscr{X}$ be a subset of 2^X with $\emptyset \in \mathscr{X}$ and $X \in \mathscr{X}$. Then, $(X, \mathscr{X})$ is called a fuzzy measurable space. We say that a function $f : X \to \mathbb{R}^+$ is $\mathscr{X}$-measurable if $\{x|f(x) \geq a\} \in \mathscr{X}$ for all a.*

Definition 2. *[3] Let f and g be $\mathscr{X}$-measurable functions on X; then, we say that f and g are comonotonic if*

$$f(x) < f(y) \Rightarrow g(x) \leq g(y)$$

for $x, y \in X$.

Definition 3. *[15] Let $(X, \mathscr{X})$ be a fuzzy measurable space; then, a fuzzy measure μ on $(X, \mathscr{X})$ is a real valued set function, $\mu : \mathscr{X} \longrightarrow \mathbb{R}^+$ with the following properties.*

(i) $\mu(\emptyset) = 0$, $\mu(X) = k$ *where* $k \in (0, \infty)$.
(ii) $\mu(A) \leq \mu(B)$ *whenever* $A \subseteq B$, $A, B \in \mathscr{X}$.

A triplet $(X, \mathscr{X}, \mu)$ is said to be a fuzzy measure space.

Definition 4. *Let μ be a fuzzy measure on $(X, \mathscr{X})$, we say that:*

(i) *μ is a one to one fuzzy measure, if $A \neq B$ implies $\mu(A) \neq \mu(B)$;*
(ii) *μ is a distorted additive measure if there exist a strictly monotone function f and an additive measure m such that $\mu = f \circ m$.*

Definition 5. *[2,12] Let $(X, \mathscr{X}, \mu)$ be a fuzzy measure space and let f be a $\mathscr{X}$-measurable function; then, the Choquet integral of f with respect to μ is defined by*

$$(C) \int f d\mu := \int_0^\infty \mu_f(r) dr,$$

where $\mu_f(r) = \mu(\{x|f(x) \geq r\})$.

Definition 6. *[1] For any $r > 0$ and $A \in \mathscr{X}$, the basic simple function $b(r,A)$ is defined by $b(r,A)(x) = r$ if $x \in A$ and $b(r,A)(x) = 0$ if $x \notin A$.*

Then, we say that a function f is a simple function if it can be expressed as

$$f := \sum_{i=1}^{n} b(a_i, A_i) \text{ for } a_i > 0 \tag{1}$$

where $A_1 \supsetneq A_2 \supsetneq \cdots \supsetneq A_n$, $A_i \in \mathscr{X}$.

Expression 1 is called a comonotonic additive representation of f. f can also be expressed as $f := \vee_{i=1}^{n} b(a'_i, A_i)$ for $a'_1 > \cdots > a'_n > 0$, where $A_1 \supsetneq A_2 \supsetneq \cdots \supsetneq A_n$, $A_i \in \mathscr{X}$. This expression is called a comonotonic maxitive representation of f.

Then, when a $\mathscr{X}$-measurable function f is a simple function with a comonotonic additive representation, we have that the following equation holds:

$$(C)\int f d\mu = \sum_{i=1}^{n} a_i \mu(A_i).$$

Definition 7. *[14,15] Let $(X, \mathscr{X}, \mu)$ be a fuzzy measure space and let $f : X \to [0,\infty)$ be a $\mathscr{X}$-measurable function; then, the Sugeno integral of f with respect to μ is defined by*

$$(S)\int f d\mu := \sup_{r\in[0,\infty)} [r \wedge \mu_f(r)].$$

When f is a simple function with a comonotonic maxitive representation, the Sugeno integral can be written as

$$(S)\int f d\mu = \bigvee_{i=1}^{n} (a_i' \wedge \mu(A_i)).$$

2.2 Generalized Fuzzy Integral

In this section, we define a generalized fuzzy integral in terms of a pseudo-addition $\oplus$ and a pseudo-multiplication $\boxdot$. Formally, $\oplus$ and $\boxdot$ are binary operators that generalize addition and multiplication, and also max and min. We want to recall that generalized fuzzy integrals have been investigated by Benvenuti et al. in [1].

Note that we will use $k \in (0,\infty)$ in the rest of this paper.

Definition 8. *A pseudo-addition $\oplus$ is a binary operation on $[0,k]$ or $[0,\infty)$ fulfilling the following conditions:*

(A1) $x \oplus 0 = 0 \oplus x = x$.
(A2) $x \oplus y \le u \oplus v$ *whenever* $x \le u$ *and* $y \le v$.
(A3) $x \oplus y = y \oplus x$.
(A4) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$.
(A5) $x_n \to x, y_n \to y$ *implies* $x_n \oplus y_n \to x \oplus y$.

A pseudo-addition $\oplus$ is said to be strict if and only if $x \oplus y < x \oplus z$ whenever $x > 0$ and $y < z$, for $x, y, z \in (0,k)$; and it is said to be Archimedean if and only if $x \oplus x > x$ for all $x \in (0,k)$.

Definition 9. *A pseudo-multiplication $\boxdot$ is a binary operation on $[0,k]$ or $[0,\infty)$ fulfilling the conditions:*

(M1) *There exists a unit element $e \in (0,k]$ such that $x \boxdot e = e \boxdot x = x$.*
(M2) $x \boxdot y \le u \boxdot v$ *whenever* $x \le u$ *and* $y \le v$.
(M3) $x \boxdot y = y \boxdot x$.
(M4) $(x \boxdot y) \boxdot z = x \boxdot (y \boxdot z)$.
(M5) $x_n \to x, y_n \to y$ *implies* $x_n \boxdot y_n \to x \boxdot y$.

Example 1

(i) The maximum operator $x \vee y$ is a non Archimedean pseudo-addition on $[0,k]$.
(ii) The sum $x+y$ is an Archimedean pseudo-addition on $[0,\infty)$.
(iii) The Sugeno operator $x +_\lambda y := 1 \wedge (x+y+\lambda xy)$ $(-1 < \lambda < \infty)$ is an Archimedean pseudo-addition on $[0,1]$.

Proposition 1. *[9] If a pseudo-addition $\oplus$ is Archimedean, then there exists a continuous and strictly increasing function $g : [0,k] \to [0,\infty]$ such that $x \oplus y = g^{(-1)}(g(x) + g(y))$, where $g^{(-1)}$ is the pseudo-inverse of g defined by*

$$g^{(-1)}(u) := \begin{cases} g^{(-1)}(u) & \text{if } u \leq g(k) \\ k & \text{if } u > g(k). \end{cases}$$

The function g is called an additive generator of $\oplus$.

Definition 10. *Let μ be a fuzzy measure on a fuzzy measurable space $(X, \mathscr{X})$; then, we say that μ is a $\oplus$-measure or a $\oplus$-decomposable fuzzy measure if $\mu(A \cup B) = \mu(A) \oplus \mu(B)$ whenever $A \cap B = \emptyset$ for $A, B \in \mathscr{X}$.*

A $\oplus$-measure μ is called normal when either $\oplus = \vee$, or $\oplus$ is Archimedean and $g \circ \mu$ is an additive measure. Here, g corresponds to an additive generator of $\oplus$.

Definition 11. *Let $k \in (0,\infty)$, let $\oplus$ be a pseudo-addition on $[0,k]$ or $[0,\infty)$ and let $\boxdot$ be a pseudo-multiplication on $[0,k]$ or $[0,\infty)$; then, we say that $\boxdot$ is $\oplus$-fitting if*

(F1) $a \boxdot x = 0$ *implies* $a = 0$ *or* $x = 0$,
(F2) $a \boxdot (x \oplus y) = (a \boxdot x) \oplus (a \boxdot y)$.

Under these conditions, we say that $(\oplus, \boxdot)$ is a pseudo-fitting system.

Let $\oplus$ be a pseudo-addition; then, we define its pseudo-inverse $-_\oplus$ as

$$a -_\oplus b := \inf\{c | b \oplus c \geq a\}$$

for all $(a,b) \in [0,k]^2$.

Definition 12. *[16] Let μ be a fuzzy measure on a fuzzy measurable space $(X, \mathscr{X})$, and let $(\oplus, \boxdot)$ be a pseudo-fitting system. Then, when μ is a normal $\oplus$-measure, we define the pseudo-decomposable integral of a measurable simple function f on X such that $f = \oplus_{i=1}^{n} b(r_i, D_i)$ where $D_i \cap D_j \neq \emptyset$ for $i \neq j$, as follows:*

$$(D) \int f d\mu := \oplus_{i=1}^{n} r_i \boxdot \mu(D_i).$$

Since μ is an $\oplus$-measure, it is obvious that the integral is well defined.

Definition 13. *Let μ be a fuzzy measure on a measurable space $(X, \mathscr{X})$, and let $(\oplus, \boxdot)$ be a pseudo-fitting system. Then, the generalized fuzzy integral (GF-integral) of a measurable simple function $f := \oplus_{i=1}^{n} b(a_i, A_i)$, with $a_i > 0$ and $A_1 \supsetneq A_2 \supsetneq \ldots A_n$, $A_i \in \mathscr{X}$, is defined as follows:*

$$(GF) \int f d\mu := \oplus_{i=1}^{n} a_i \boxdot \mu(A_i).$$

The GF-integral of a simple function is well defined [1].

The next proposition follows from the definition of the pseudo-inverse $-_{\oplus}$, the generalized t-conorm integral (Definition 13), and the t-conorm integral (Definition 12).

Proposition 2. *Let μ be a fuzzy measure on a fuzzy measurable space $(X, \mathscr{X})$, and let $(\oplus, \boxdot)$ be a pseudo-fitting system. Then, if μ is a normal $\oplus$-measure, the generalized fuzzy integral coincides with the pseudo-decomposable integral.*

Example 2

(i) When $\oplus = +$ and $\boxdot = \cdot$, the generalized fuzzy integral is a Choquet integral.
(ii) When $\oplus = \vee$ and $\boxdot = \wedge$, the generalized fuzzy integral is a Sugeno integral.

Let f, g be comonotonic measurable functions. Then, since for all $a, b > 0$ either $\{x|f(x) \geq a\} \subseteq \{x|g(x) \geq b\}$ or $\{x|f(x) \geq a\} \supseteq \{x|g(x) \geq b\}$, the following theorem can be proved.

Theorem 1. *[13] Let $(X, \mathscr{X}, \mu)$ be a fuzzy measure space and let $(\oplus, \boxdot)$ be a pseudo-fitting system. Then, for comonotonic measurable functions f, and g, we have*

$$(GF)\int (f \oplus g) d\mu = (GF)\int f d\mu \oplus (GF)\int g d\mu.$$

We call this property the comonotonic $\oplus$-additivity of a generalized fuzzy integral.

2.3 Multisets

Let X be a universal set. Then, a multiset M over X is characterized by the count function $C_M : X \to N := \{0, 1, 2, \dots\}$, where C_M corresponds to the number of occurrences of the object $x \in X$.

We denote by $\mathscr{M}(X)$ the class of multisets of X.

Example 3. *Let $X := \{a, b, c\}$ and $M := \{a, a, a, b, b\}$. Then $C_M(a) = 3$, $C_M(b) = 2$, $C_M(c) = 0$.*

The multiset M in Example 3 can also be represented as $M = \{3/a, 2/b\}$ or $M = \{(a, 3), (b, 2)\}$.

Definition 14. *Let $M, N \in \mathscr{M}(X)$. Then, we define:*

– *the inclusion of multisets by*

$$M \subseteq N \Leftrightarrow C_M(x) \leq C_N(x)$$

for all $x \in X$;
– *the equality of multisets $M = N$ by*

$$C_M(x) = C_N(x)$$

for all $x \in X$.

Let $M \in \mathscr{M}(X)$. Then $\mathscr{P}(M)$ denotes the class of subsets of the multiset M, that is,

$$\mathscr{P}(M) := \{N | N \subseteq M, N \in \mathscr{M}(X)\}.$$

Proposition 3. *Let $|X| = n$ and $M \in \mathscr{M}(X)$. If $M = \{(a_i, k_i) | i = 1, 2, \ldots n\}$, then*

$$|\mathscr{P}(M)| = \prod_{i=1}^{n} (k_i + 1).$$

Example 4. *Let $M = \{a, a, a, b, b\}$. Then*

$$\begin{aligned}\mathscr{P}(M) = \{ \quad & M_1 = \emptyset, \quad M_2 = \{a, a\}, \ M_3 = \{a, a, a\}, \quad M_4 = \{a, a, a, b\}, \\ & M_5 = \{a\}, \ M_6 = \{a, b\}, \ M_7 = \{a, a, b\}, \quad M_8 = \{a, a, b, b\}, \\ & M_9 = \{b\}, M_{10} = \{b, b\}, M_{11} = \{a, b, b\}, M_{12} = \{a, a, a, b, b\}\}.\end{aligned}$$

Definition 15. *Let $A, B \in \mathscr{M}(X)$. We define some binary operations on $\mathscr{M}(X)$. Definitions include union, intersection and addition of two multisets.*

(i) $C_{A \cup B}(x) = C_A(x) \vee C_B(x)$
(ii) $C_{A \cap B}(x) = C_A(x) \wedge C_B(x)$
(iii) $C_{A+B}(x) = C_A(x) + C_B(x)$
(iv) $C_{A \oplus B}(x) = C_A(x) \oplus C_B(x)$
(v) $C_{A \boxdot B}(x) = C_A(x) \boxdot C_B(x)$

where $x \in X$ and C_A is a count function of A.

Proposition 4. *Let $A, B \in \mathscr{M}(X)$. We have*

$$A \cap B \subseteq A \cup B \subseteq A + B$$

Example 5. *Let $X := \{a, b, c\}$ and $A := \{a, a, b\}$, $B := \{a, b, b, c\}$. Then we have*

(i) $A \cup B = \{a, a, b, b, c\}$
(ii) $A \cap B = \{a, b\}$
(iii) $A + B = \{a, a, a, b, b, b, c\}$
(iv) $A \oplus B = \{a, a, b, b, c\}$ *when* $\oplus = \vee$
(v) $A \boxdot B = \{a, b\}$ *when* $\oplus = \wedge$

3 Representation of Fuzzy Measures for Finite Multisets

Let X be a finite universal set, $|X| = n$, and $\mathbb{P}$ be the set of prime numbers, that is, $\mathbb{P} := \{2, 3, 5, 7, \ldots\}$. Since X is a finite set, there exists a one to one mapping φ_X from X to a subset of $\mathbb{P}$. That is,

$$\varphi_X : X \to \{p_1, p_2, \ldots, p_n\}.$$

Let $M \in \mathscr{M}(X)$, then, we have an induced one to one mapping Φ_X from $\mathscr{M}(X)$ to a subset S of natural numbers by

$$\Phi_X(M) := \prod_{i=1}^{n} \varphi_X(x_i)^{C_M(x_i)}.$$

We say that $\Phi_X(M)$ is a natural number representation of the multiset M.

Example 6. *Let $X := \{a,b,c\}$ and $\varphi_X(a) = 2, \varphi_X(b) = 3, \varphi_X(c) = 5$. Then,*

$$\Phi_X(A) := 2^{C_A(a)} 3^{C_A(b)} 5^{C_A(c)}$$

for $A \in \mathscr{M}(X)$. For example, if $A := \{a,a,b,c\}$, then

$$\Phi_X(A) = 2^2 \cdot 3 \cdot 5 = 60.$$

Let $M \in \mathscr{M}(X)$. We have

$$\Phi_X(\mathscr{P}(M)) := \left\{ \prod_{i=1}^{n} \varphi_X(x_i)^{C_A(x_i)} \mid \ A \in \mathscr{P}(M) \right\}.$$

Proposition 5. *Let $M \in \mathscr{M}(X)$; then, $\Phi_X(\mathscr{P}(M))$ is the set of divisors of $\Phi_X(M)$.*

Example 7. *Let $X := \{a,b,c\}$, let $\varphi_X(a) = 2, \varphi_X(b) = 3, \varphi_X(c) = 5$, and let the multiset M be defined by $M := \{a,a,a,b,b,c\}$. Then, $\Phi_X(M) := 2^3 3^2 5^1 = 120$, and*

$$\Phi_X(\mathscr{P}(M)) := \{1,2,3,4,5,6,8,10,12,15,24,30,40,60,120\}.$$

Definition 16. *[18] Let X be a reference set, let M be a multiset on X such that $M \neq \emptyset$; then, a function μ from $(M, \mathscr{P}(M))$ to $[0,1]$ is a fuzzy measure if the following holds:*

- *$\mu(\emptyset) = 0$ and $\mu(M) = k$ for $k \in (0,\infty)$,*
- *$\mu(A) \leq \mu(B)$ when $A \subseteq B$ and $B \subseteq M$.*

We have the next proposition.

Proposition 6. *Let Φ_X be a natural number representation of multisets. Then, for any non-decreasing function such that $\rho(1) = 0$, and $\rho(\Phi_X(M)) > 0$, $\rho \circ \Phi_X$ is a fuzzy measure.*

Example 8. *Let Φ_X be a natural number representation of multisets. Then we have that $\mu := \rho \circ \Phi_X$ with $\rho(x) = \log(x)$ is a fuzzy measure.*

Conversely, we can represent any fuzzy measure in terms of a natural number representation Φ_X and a distortion function.

Proposition 7. *Let μ be a one to one fuzzy measure. There exists a function $f_\mu : \mathbb{R}^+ \to \mathbb{R}$ such that $\mu = f_\mu \circ \Phi_X$.*

In this case, we say that f_μ is a representation function of a one to one fuzzy measure.

Proposition 8. *Let μ be a one to one fuzzy measure. If a representation function f_μ is monotone, then μ is a distorted additive measure.*

The representations of fuzzy measures in terms of mappings on the set of prime numbers are further studied in [18].

4 Comonotonicity of Multisets

We start this section with a definition of the concept of comonotonicity of multisets.

Definition 17. *Let M,N be multisets on X. Then, we say that M and N are comonotonic if $C_M(x_1) < C_M(x_2)$ implies $C_N(x_1) \leq C_N(x_2)$ for all pairs $x_1, x_2 \in X$.*

Let M,N be two universal sets. Then, since $C_M(x) = 0$ or 1 and $C_N(x) = 0$ or 1, M and N are comonotonic if and only if $M \subseteq N$ or $M \supseteq N$. However, in general there exist comonotonic multisets $M,N \in \mathscr{M}(X)$ such that $M \not\subseteq N$ and $M \not\supseteq N$. Indeed, a multiset M which satisfies $C_M(x_i) = C_M(x_j)$ for all $x_i, x_j \in X$ is comonotonic with any multiset $N \in \mathscr{M}(X)$. In particular, the emptyset is comonotonic with any set $N \in \mathscr{M}(X)$. Finally, $M \subseteq N$ does not imply the comonotonicity of M and N.

Proposition 9. *Comonotonicity of multisets is a reflexive and symmetric binary relation. However, it is not an equivalence relation, since it is not transitive.*

Example 9. *Let $X := \{a,b,c\}$ and consider the natural number representation of multisets Φ_X induced by $\varphi_X(X) = \{2,3,5\}$. Let*

$$\begin{array}{ll} M_1 := \{a,a,a,b,b,c\}, & \Phi_X(M_1) = 360; \\ M_2 := \{a,a,b,c\}, & \Phi_X(M_2) = 60; \\ M_3 := \{a,b,c\}, & \Phi_X(M_3) = 30; \\ M_4 := \{c,c,c\}, & \Phi_X(M_4) = 125, \\ M_5 := \{b,b,b,b,c,c,c\}, & \Phi_X(M_5) = 10125. \end{array}$$

Then M_1 and M_2 are comonotonic. We have that $M_3 \not\subseteq M_4$ and $M_3 \not\supseteq M_4$, but M_3 and M_4 are comonotonic anyway. If comonotonicity had been an equivalence relation, then the fact that M_2 and M_3 are comonotonic and the transitivity of the equivalence relation would have implied the comonotonicity of M_2 and M_4. However M_2 and M_4 are not comonotonic, hence illustrating the abscence of transitivity. Finally $M_4 \subseteq M_5$ but M_4 and M_5 are not comonotonic.

Let M be a multiset and $\mathscr{P}(M)$ be the class of submultisets of M. Because of the symmetry of the comonotonicity, we can decompose $\mathscr{P}(M)$ in blocks of pairwise comonotonic multisets, as will be described in the following.

Example 10. *Let $X := \{a,b\}$ and $M := \{a,a,b\}$. We can decompose $\mathscr{P}(M)$ in blocks of pairwise comonotonic multisets as*

$$\mathscr{P}(M) = M_1 \cup M_2,$$

with

- $M_1 := \{\emptyset, \{a\}, \{a,a\}, \{a,b\}, \{a,a,b\}\}$ *and*
- $M_2 := \{\emptyset, \{b\}, \{a,b\}\}$.

Then the members of each M_i (i=1,2) are pairwise comonotonic. Observe that $M_1 \cap M_2 \neq \emptyset$.

Consider the natural number representation of multisets Φ_X induced by the function $\varphi_X(X) := \{2,3\}$. Then $\Phi_X(M) = 2^2 3 = 12$ and we have

$$\Phi_X(\mathscr{P}(M)) = \Phi_X(M_1) \cup \Phi_X(M_2),$$

with

- $M_1 := \{1,2,4,6,12\}$ *and*
- $M_2 := \{1,3,6\}$.

Lemma 1. *Let $\mathscr{F} \subseteq \mathscr{M}(X)$ be a family of multisets over X. Then, there exist sets (which we call blocks) M_i $i = 1,2,\ldots,k$ of elements of $\mathscr{F}$, such that $\mathscr{F} = \cup_{1 \leq i \leq k} M_i$ and such that the members of a block M_i are pairwise comonotonic. Also, these blocks can always be chosen to be maximal with respect to inclusion.*

We say that a decomposition of $\mathscr{F}$ in comonotonic blocks, which is maximal with respect to inclusion, is a maximal comonotonic block decomposition of $\mathscr{F}$ and we call each M_i a comonotonic block of $\mathscr{F}$. If $\mathscr{F} = \mathscr{P}(M)$ for some multiset M over X, then we abuse notation and talk about the block decomposition of M and the comonotonic blocks of M.

Let M_1 and M_2 be two comonotonic blocks of $\mathscr{M}(X)$. We say that M_1 and M_2 are different if $M_1 \triangle M_2 \neq \emptyset$ where $M_1 \triangle M_2 := (M_1 \cap M_2^c) \cup (M_2 \cap M_1^c)$ and where M^c is the standard complement.

Definition 18. *Let $\mathscr{F} \subseteq \mathscr{M}(X)$ be a family of multisets over X and let $\mathscr{F} = \cup_{1 \leq i \leq l} M_i$ be a maximal comonotonic block decomposition of $\mathscr{F}$. The positive integer l is is said to be the variety of $\mathscr{F}$.*

Suppose that $X = \{a_1, a_2, \ldots, a_n\}$. It is not hard to see that the variety of $\mathscr{M}(X)$ coincides with the number of permutations of $C_M(a_1), C_M(a_2), \ldots, C_M(a_n)$, proving the next proposition.

Proposition 10. *Suppose that $|X| = n$. The variety of $\mathscr{M}(X)$ is $n!$.*

Example 11. *Let $X := \{a,b,c\}$. Proposition 10 says that the variety of $\mathscr{M}(X)$ is 6. Indeed the maximal comonotonic block decomposition of $\mathscr{M}(X)$ is*

$$\mathscr{M}(X) = \bigcup_{i=1}^{6} M_i$$

with

$$M_1 := \{M | C_M(a) \leq C_M(b) \leq C_M(c)\}$$
$$M_2 := \{M | C_M(a) \leq C_M(c) \leq C_M(b)\}$$
$$M_3 := \{M | C_M(b) \leq C_M(a) \leq C_M(c)\}$$
$$M_4 := \{M | C_M(b) \leq C_M(c) \leq C_M(a)\}$$

$$M_5 := \{M | C_M(c) \leq C_M(a) \leq C_M(b)\}$$
$$M_6 := \{M | C_M(c) \leq C_M(b) \leq C_M(a)\}.$$

Let $M = \{a,a,b,c\}$, then, following Section 3 we use the natural number representation of M induced by the function $\varphi_X(\{a,b,c\}) = \{2,3,5\}$, and we get

$$M_1 \cap \mathscr{P}(M) := \{\emptyset, \{c\}, \{b,c\}, \{a,b,c\}\} = \{1,5,15,30\}$$
$$M_2 \cap \mathscr{P}(M) := \{\emptyset, \{b\}, \{b,c\}, \{a,b,c\}\} = \{1,3,15,30\}$$
$$M_3 \cap \mathscr{P}(M) := \{\emptyset, \{c\}, \{a,c\}, \{a,b,c\}\} = \{1,5,10,30\}$$
$$M_4 \cap \mathscr{P}(M) := \{\emptyset, \{a\}, \{a,a\}, \{a,c\}, \{a,a,c\}, \{a,b,c\}, \{a,a,b,c\}\} = \{1,2,4,10,20,30,60\}$$
$$M_5 \cap \mathscr{P}(M) := \{\emptyset, \{b\}, \{a,b\}, \{a,b,c\}\} = \{1,3,6,30\}$$
$$M_6 \cap \mathscr{P}(M) := \{\emptyset, \{a\}, \{a,a\}, \{a,b\}, \{a,a,b\}, \{a,b,c\}, \{a,a,b,c\}\} = \{1,2,4,6,12,30,60\}.$$

5 Extension of Fuzzy Measure to Multisets

Even if X is a finite set, $\mathscr{M}(X)$ is infinite. The problem is how to define a fuzzy measure on $\mathscr{M}(X)$. We give a partial solution to this problem.

If X is a finite set, then 2^X is also a finite set. We can define a fuzzy measure μ on 2^X.

Definition 19. *Let M be a multiset on X, then we can represent a count function C_M by $C_M := \oplus_{i=1}^n b(a_i, A_i)$, with $a_i > 0$ and $A_1 \supsetneq A_2 \supsetneq \ldots A_n$, $A_i \in \mathscr{X}$. Then we can define an extension $\bar{\mu}$ of the fuzzy measure μ to a multiset M by*

$$\bar{\mu}(M) := \oplus_{i=1}^n a_i \boxdot \mu(A_i) = (GF) \int C_M d\mu$$

with $a_i > 0$ and $A_1 \supsetneq A_2 \supsetneq \ldots A_n$, $A_i \in \mathscr{X}$.
We say that $\bar{\mu}$ is a comonotonic $(\oplus, \boxdot)$-extension of μ.

As a corollary of Theorem 1 we get the following proposition.

Proposition 11. *Let $M, N \in \mathscr{M}(X)$, and let μ be a fuzzy measure on 2^X. If M and N are comonotonic, then $\bar{\mu}(M \oplus N) = \bar{\mu}(M) \oplus \bar{\mu}(N)$.*

Given a fuzzy measure ν on $\mathscr{M}(X)$ we have the next theorem, which can be seen as the converse of Definition 19.

Theorem 2. *If ν on $\mathscr{M}(X)$ is comonotonic $\oplus$-additive, then there exists a fuzzy measure μ on 2^X such that*

$$\nu(M) = (GF) \int C_M d\mu$$

for $M \in \mathscr{M}(X)$.

6 Conclusion

In this paper we have extended fuzzy measures on multisets. We have also introduced comonotonicity of multisets and given some examples.

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