The Equidistant Dimension of Corona Product Graphs*

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1 Introduction

Distance-based parameters have long played a central role in graph theory, providing fundamental tools for analyzing the structural and algorithmic properties of networks. Motivated by recent applications in privacy and anonymization [8], the equidistant dimension was introduced as an alternative perspective, focusing on the role of vertices that are equidistant from a common vertex [4]. For a connected graph G, a set $S \subseteq V(G)$ is a *distance-equalizer set* if every pair of different vertices outside S has a vertex in S equidistant from both; the minimum cardinality of such sets defines the *equidistant dimension* $\xi(G)$.

This parameter has been studied from both structural and computational perspectives, with general bounds in terms of classical invariants and exact values for families such as paths, cycles, and complete multipartite graphs [4]. Its behavior under graph products has also attracted attention, particularly for lexicographic and Cartesian products [2, 3, 6].

In this work, we study the equidistant dimension of corona product graphs. Given graphs G and H, the corona product $G \odot H$ is obtained by attaching a copy of H to each vertex of G [5]. To support our analysis, we introduce the empty bisector graph, an auxiliary structure that captures the absence of equidistant vertices and plays a central role in our results.

2 Preliminaries

Roughly speaking, a bisector is an object that divides a structure into two balanced parts. Given a graph G and two distinct vertices $u, v \in V(G)$, the bisector of u and v, denoted by $B_G(u \mid v)$, is the set of vertices of G equidistant from u and v.

Definition 1 (Empty bisector graph)**.** *Given any graph $G = (V, E)$, we define its* empty bisector graph $\widehat{G} = (V, \widehat{E})$*, where $\{u, v\} \in \widehat{E}$ if and only if $B_G(u \mid v) = \varnothing$.*

We recall the following classical result, which will be used extensively in our analysis.

Theorem 2 (Gallai's Theorem [1])**.** *For any graph G, it holds that $\alpha(G) + \beta(G) = \mathrm{n}(G)$.*

Here $\alpha(G)$ and $\beta(G)$ denote the independence number and the vertex cover number of G, respectively.

In what follows, for any graph G, we denote by $\mathcal{B}(G)$ the set of all vertex covers of G. For every integer $n \geq 1$, we denote $[n] = \{1, 2, \ldots, n\}$. From now on, we take $V(G) = \{v_1, \ldots, v_n\}$ and $H_i = (V_i, E_i)$

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as the i-th copy of H in $G \odot H$. Note that for each $i \in [n]$, all the vertices of H_i are adjacent to v_i. As usual, we denote by N_n, P_n, C_n, K_n, and $K_{r,n-r}$ the empty graphs, the path graphs, the cycle graphs, the complete graphs, and the complete bipartite graphs of order n, respectively. We denote by $K_{n_1,\ldots,n_p}$ the *complete p-partite graphs*, with vertex set $A_1 \cup \cdots \cup A_p$ such that $A_1, \ldots, A_p$, are pairwise disjoint, verifying $|A_i| = n_i \geq 1$. Finally the *bistar graphs* are denoted by $K_2(r, s)$.

All graphs considered in this work are undirected, simple, and finite.

Lemma 3. *Let G and H be two graphs, with G connected. For every distance-equalizer set S of $G \odot H$ we have*

$$S \cap (\{v_i\} \cup V(H_i)) \neq \varnothing, \text{ for all } i \in [\mathrm{n}(G)],$$

and as a consequence, $\xi(G \odot H) \geq \mathrm{n}(G)$

Proof. If $\mathrm{n}(G) = 1$, the result is immediate, as $V(G \odot H) = \{v_1\} \cup V(H_1)$. Otherwise, let $S \subseteq V(G \odot H)$ be a distance-equalizer set of $G \odot H$. Since for any $i \in [\mathrm{n}(G)], h \in V(H_i)$ and $u \in V(G \odot H) \setminus (\{v_i\} \cup V(H_i))$ we have $d(u, h) = d(u, v_i) + 1$, it follows that $S \cap (\{v_i\} \cup V(H_i)) \neq \varnothing$. □

Definition 4. *Let G and H be two graphs, with G connected. For any set $S \subseteq V(G \odot H)$, we define the following two operators:*

(i) $L(S) = S \cap V(G)$,

(ii) $U(S) = \{v_i \in V(G) : V(H_i) \cap S \neq \varnothing\}$.

According to the definition of $\widehat{G}$, for any pair of vertices $v_i, v_j \in V(G)$ satisfying $\{v_i, v_j\} \in E(\widehat{G})$, we have $B(v_i \mid v_j) = \varnothing$. This leads to the following result.

Lemma 5. *Let G and H be two graphs, with G connected. If S is a distance-equalizer set of $G \odot H$, then $L(S) \in \mathcal{B}(\widehat{G})$, $U(S) \in \mathcal{B}(\widehat{G})$, and $L(S) \cup U(S) = V(G)$.*

Observe that, $|L(S)| \geq \beta(\widehat{G})$ and $|U(S)| \geq \beta(\widehat{G})$, since $L(S)$ and $U(S)$ are vertex covers of $\widehat{G}$.

The next definition plays a key role in the characterization of distance-equalizer sets satisfying certain convenient conditions in corona product graphs.

Definition 6. *Let G be a graph and let $X, Y \subseteq V(G)$ such that $X \cup Y = V(G)$. We say that (X, Y) is a* forward-equalized pair of G*, if for every pair of vertices $(u, v) \in (X \setminus Y) \times (Y \setminus X)$, there exists a vertex $w \in V(G)$ such that $d_G(w, u) = d_G(w, v) + 1$.*

3 Main results

In this section, we present the main general results of our analysis, obtained using the framework of the empty bisector graph. We begin by characterizing distance-equalizer sets in corona product graphs.

Theorem 7. *Let G, H be two graphs, with G connected, and let X, Y be two vertex covers of $\widehat{G}$. The set*

$$S = \left(\bigcup_{v_i \in X} V(H_i) \right) \cup Y$$

is a distance-equalizer set of $G \odot H$, if and only if, (X, Y) is a forward-equalized pair of G.

Sketch of proof. The necessity follows by applying Lemma 5 to the sets $X = U(S)$ and $Y = L(S)$, showing that (X, Y) is a forward-equalized pair. For the sufficiency, we distinguish four possible cases for any two vertices $u, v \in V(G \odot H) \setminus S$, and verify in each case that $B_{G \odot H}(u \mid v) \cap S \neq \varnothing$. □

Lemma 8. *For any two graphs G and H, with G connected, there exists a $\xi(G \odot H)$-set S, such that $V(H_i) \cap S = \varnothing$ or $V(H_i) \subseteq S$, for every $i \in [\mathrm{n}(G)]$.*

Sketch of proof. This can be proven by contradiction. Let us consider a set $S \subseteq V(G \odot H)$ such that $|S \setminus V(G)|$ is minimum among all $\xi(G \odot H)$-set. If there exist $u \in V(H_i) \cap S$ and $w \in V(H_i) \setminus S$, for some $i \in [\mathrm{n}(G)]$, then it can be proved that the set $S' = (S \cup \{v_i\}) \setminus \{u\}$ is a is a distance-equalizer set of $G \odot H$. In contradiction with the minimality assumption on S or the fact that S is a $\xi(G \odot H)$-set. □

Lemma 8 implies there always exists a $\xi(G \odot H)$-set which is independent of the structure of the graph H. Hence, the only property of H that is relevant for determining $\xi(G \odot H)$ is the order.

We introduce the following parameter, which is used to establish a sharp lower bound for the equidistant dimension of corona product graphs.

Definition 9. *For any connected graph G, we define*

$$\beta^*(G) = \min_{X,Y \in \mathcal{B}(\widehat{G})} \{|X \cap Y| : (X, Y) \text{ is a forward-equalized pair of } G\}$$

The following two results establish general sharp bounds for the equidistant dimension a corona product graphs.

Theorem 10. *For any two graphs G and H, with G connected,*

$$\xi(G \odot H) \geq \beta(\widehat{G})\, \mathrm{n}(H) + \alpha(\widehat{G}) + \beta^*(G).$$

Sketch of proof. We consider a $\xi(G \odot H)$-set S which satisfies the conditions of Lemma 8. Applying Theorem 7, it follows that $(U(S), L(S))$ is a forward-equalized pair of G. Since $|U(S) \cap L(S)| \geq \beta^*(G)$, the result can be derived from the expression $|S| = |U(S)|\, \mathrm{n}(H) + |L(S)|$. □

Theorem 11. *If G and H are two graphs such that G is connected, then*

$$\xi(G \odot H) \leq \beta(\widehat{G})\, \mathrm{n}(H) + \mathrm{n}(G).$$

Proof. Let U be a $\beta(\widehat{G})$- set, and let $L = V(G)$. Observe that U and L are vertex covers of $\widehat{G}$ and (U, L) is a forward-equalized pair of G. Consequently, by Theorem 7 the set

$$S = \left(\bigcup_{v_i \in U} V(H_i) \right) \cup L$$

is a distance-equalizer set of $G \odot H$. Now we have, $\xi(G \odot H) \leq |S| = \beta(\widehat{G})\, \mathrm{n}(H) + \mathrm{n}(G)$, which proves the claim. □

The lower and upper bounds stated in Theorems 10 and 11 are tight. Several examples in [7] illustrate cases in which these bounds are attained.

Theorem 12. *If G is a connected graph and H any graph, then $\xi(G \odot H) = \mathrm{n}(G)$ if and only if either $\beta(\widehat{G}) = 0$, or ($\mathrm{n}(H) = 1$ and $\beta^*(G) = 0$).*

Sketch of proof. Assume first that $\xi(G \odot H) = \mathrm{n}(G)$. Combining Theorem 10 with Gallai's Theorem 2, we obtain $\beta(\widehat{G})(\mathrm{n}(H) - 1) + \beta^*(G) \leq 0$. Hence, by nonnegativity of the parameters and the inequality $\beta^*(G) \leq \beta(\widehat{G})$, we have either $\beta(\widehat{G}) = 0$, or $\mathrm{n}(H) = 1$ and $\beta^*(G) = 0$.

Conversely, Lemma 3 ensures $\xi(G \odot H) \geq \mathrm{n}(G)$. If $\beta(\widehat{G}) = 0$, the upper bound in Theorem 11 yields equality. Otherwise, when $\mathrm{n}(H) = 1$ and $\beta^*(G) = 0$, a forward-equalized pair covering $V(G)$, which can be taken to be disjoint, produces a distance-equalizer set of size $\mathrm{n}(G)$, via Theorem 7. Therefore, in both cases, $\xi(G \odot H) = \mathrm{n}(G)$. □

Lemma 13. *If G is a bipartite connected graph, then the corresponding empty bisector graph $\widehat{G}$ is complete bipartite.*

Sketch of proof. Let $u, v \in V(G)$ be distinct vertices. Since G is bipartite, u and v belong to the same partite set if and only if every path between them has even length. This implies that $B_G(u \mid v) = \varnothing$ if and only if u and v belong to the same partite set of G. Therefore, the result follows. □

Theorem 14. *If G is a connected bipartite graph, with partite sets A, B such that $|A| \geq |B|$, then for any graph H,*

$$\xi(G \odot H) = |B|\,\mathrm{n}(H) + |A|.$$

Sketch of proof. By Lemma 13, $\widehat{G}$ is a complete bipartite graph. Hence, A forms an $\alpha(\widehat{G})$-set and B a $\beta(\widehat{G})$-set. Applying Theorem 7, a set S with $U(S) = B$ and $L(S) = A$, can be constructed as a distance-equalizer set of $G \odot H$. Hence, $\xi(G \odot H) \leq |S| = |B|\,\mathrm{n}(H) + |A|$. Finally, the lower bound from Theorem 10 yields the stated equality. □

A comparison of the lower and upper bounds in Theorems 10 and 11 reveals that the equidistant dimension of the corona product $G \odot H$ grows asymptotically linearly with $\mathrm{n}(H)$, with asymptotic growth rate given by $\beta(\widehat{G})$. Now, we show that there exists a threshold beyond which the equidistant dimension of $G \odot H$ is actually linear in $n(H)$, with slope $\beta(\widehat{G})$.

Lemma 15. *Let G and H be two graphs, with G connected, and let S be a $\xi(G \odot H)$-set satisfying the conditions of Lemma 8. If $\mathrm{n}(H) \geq 2$, then*

$$|U(S)| \leq \beta(\widehat{G}) \; + \; \min\left\{ \frac{\alpha(\widehat{G})}{\mathrm{n}(H)},\; \frac{\beta(\widehat{G}) - \beta^*(G)}{\mathrm{n}(H) - 1} \right\}.$$

Theorem 16. *For any connected graph G there exists a number $k(G) \in \mathbb{N}$, with $\alpha(\widehat{G}) + \beta^*(\widehat{G}) \leq k(G) \leq \mathrm{n}(G)$, such that*

$$\xi(G \odot H) = \beta(\widehat{G})\;\mathrm{n}(H) + k(G),$$

for any graph H satisfying $\mathrm{n}(H) > \min\{\alpha(\widehat{G}), \beta(\widehat{G}) - \beta^*(G) + 1\}$, and

$$\xi(G \odot H) \leq \beta(\widehat{G})\;\mathrm{n}(H) + k(G),$$

for any graph H satisfying $\mathrm{n}(H) \leq \min\{\alpha(\widehat{G}), \beta(\widehat{G}) - \beta^*(G) + 1\}$.

The complete proof of Lemma 15 and Theorem 16 can be found in [7].

3.1 Special cases and classical families

The preceding results allow us to determine the equidistant dimension of $G \odot H$ for some some extremal cases and several classical families of graphs G.

Proposition 17. *Let G and H be a two graphs. If G contains a vertex u satisfying $\varepsilon(u) \leq 2$ and $\delta(u) = \alpha(\widehat{G})$, then*

$$\xi(G \odot H) = \beta(\widehat{G})\;\mathrm{n}(H) + \alpha(\widehat{G}) = \big(\mathrm{n}(G) - \delta(u)\big)\,\mathrm{n}(H) + \delta(u).$$

Sketch of proof. For every vertex $u \in V(G)$ satisfying $\varepsilon(u) \leq 2$, we have that the sets $L = N_G(u)$ and $U = V(G) \setminus N_G(u)$ form a bipartition of $\widehat{G}$, and (U, L) constitute a forward-equalizer set of G. In particular, if $\delta(u) = \alpha(\widehat{G})$, then the result follows. □

Proposition 18. *For every graph G, such that $\Delta(G) = \mathrm{n}(G) - 1$, and every graph H,*

$$\xi(G \odot H) = \begin{cases} \mathrm{n}(G), & \text{if } \delta(G) \geq 2 \\ \mathrm{n}(H) + \mathrm{n}(G) - 1, & \text{if } \delta(G) = 1. \end{cases}$$

Sketch of proof. Let be u an universal vertex of G. Note that $N_G(u)$ is an independent set in $\widehat{G}$. If $\delta(G) \geq 2$, then $\beta(\widehat{G}) = 0$, and applying Proposition 12, we get $\xi(G \odot H) = \mathrm{n}(G)$. Otherwise, if $\delta(G) = 1$, then $N_G(u)$ is an $\alpha(\widehat{G})$-set, and applying Proposition 17 the result follows. □

For the classical families of graphs G presented below, the equidistant dimension of $G \odot H$ can be computed applying either Theorem 14 or Proposition 12.

Proposition 19. *Let $n, r, s, p, n_1, \ldots, n_p$ be positive integers such that $n \geq 2$, $s \geq r$, $p \geq 3$, and $n_p \geq \cdots \geq n_1 \geq 1$. For every graph H, the following equalities hold:*

$$\xi(K_n \odot H) = \begin{cases} n, & \text{if } n \neq 2, \\ \mathrm{n}(H) + 1, & \text{if } n = 2. \end{cases}$$

$$\xi(K_{n_1,\ldots,n_p} \odot H) = \sum_{i=1}^{p} n_i,$$

$$\xi(P_n \odot H) = \left\lfloor \frac{n}{2} \right\rfloor \mathrm{n}(H) + \left\lceil \frac{n}{2} \right\rceil,$$

$$\xi(K_{r,s} \odot H) = r\,\mathrm{n}(H) + s,$$

$$\xi(K_2(r,s) \odot H) = r\,\mathrm{n}(H) + s,$$

$$\xi(C_n \odot H) = \begin{cases} n, & \text{if } n \text{ is odd}, \\ \dfrac{n(\mathrm{n}(H)+1)}{2}, & \text{if } n \text{ is even}. \end{cases}$$

4 Remarks and future work

Using the auxiliary empty bisector graph, we establish tight bounds for $\xi(G \odot H)$ and determine its exact value for several graph families.

Future work includes relating $\beta(\widehat{G})$ to classical invariants of G, particularly for new graph classes, and characterizing the graphs attaining the bounds in Theorems 11 and 10.

Moreover, since for fixed connected G the parameter $\xi(G \odot H)$ is asymptotically linear in $\mathrm{n}(H)$ with slope $\beta(\widehat{G})$, it is natural to study when non-isomorphic graphs have isomorphic empty bisector graphs, leading to identical asymptotic behaviour.

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