Codes and Privacy

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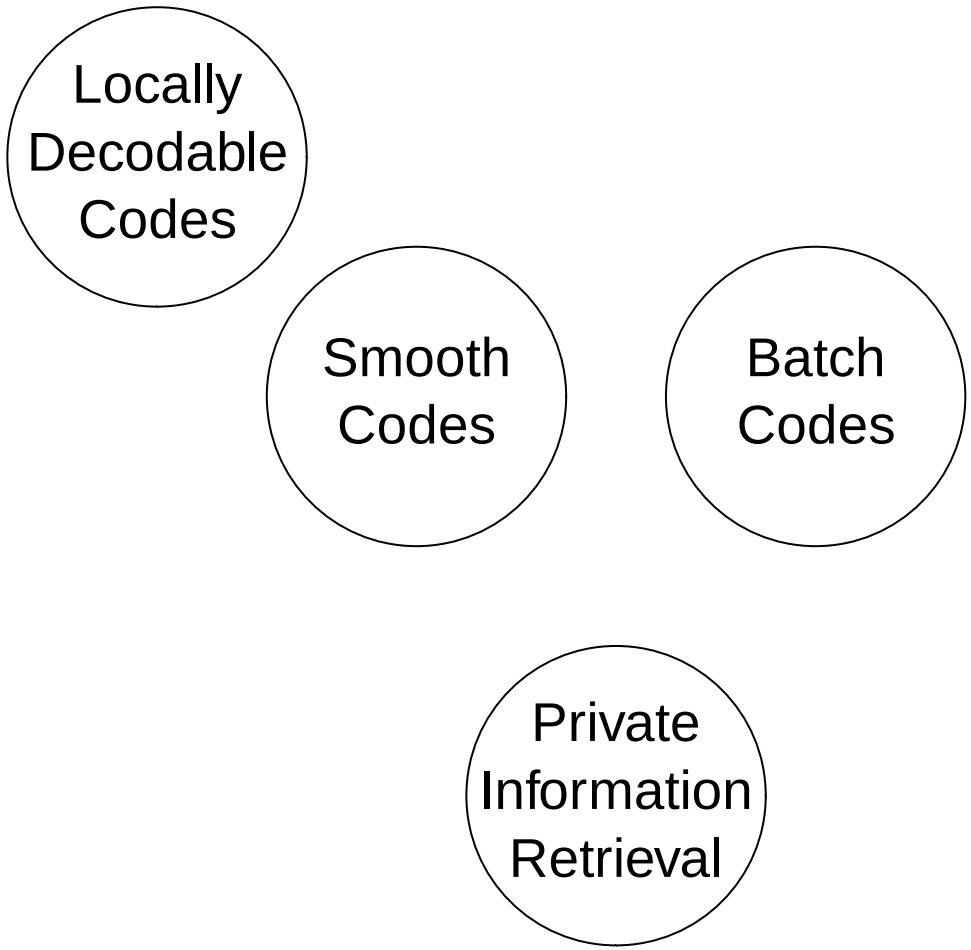
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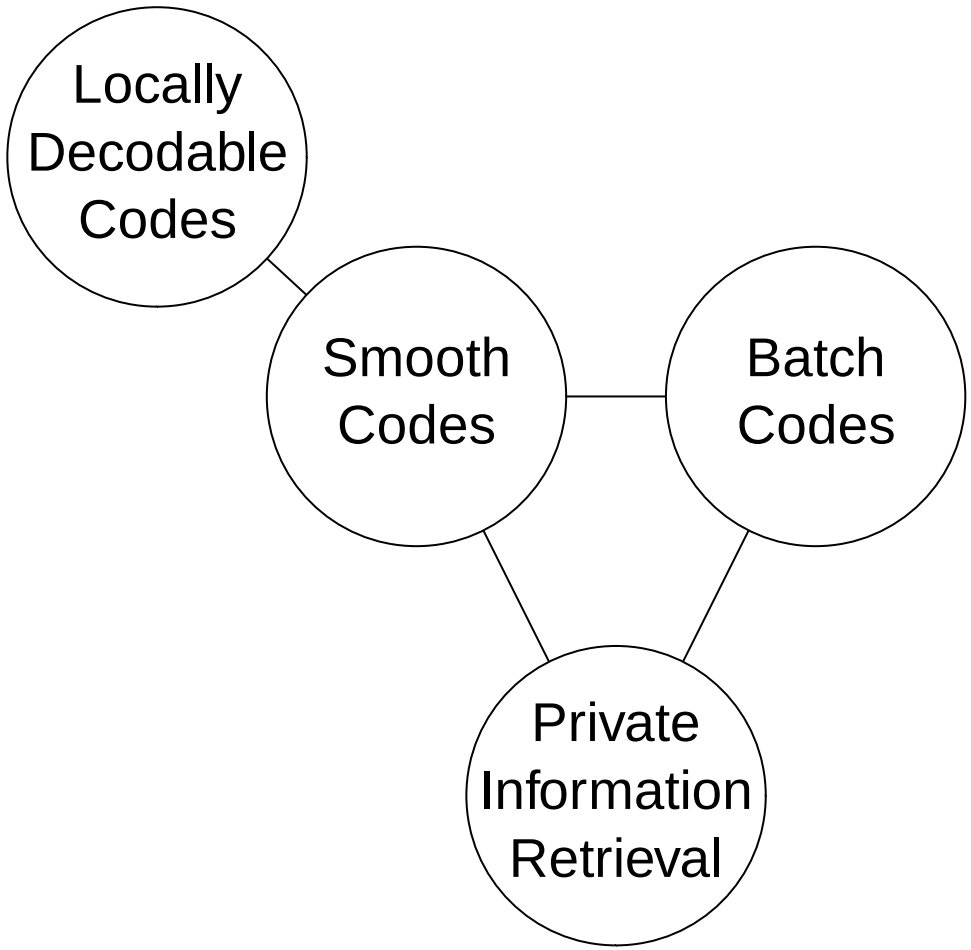
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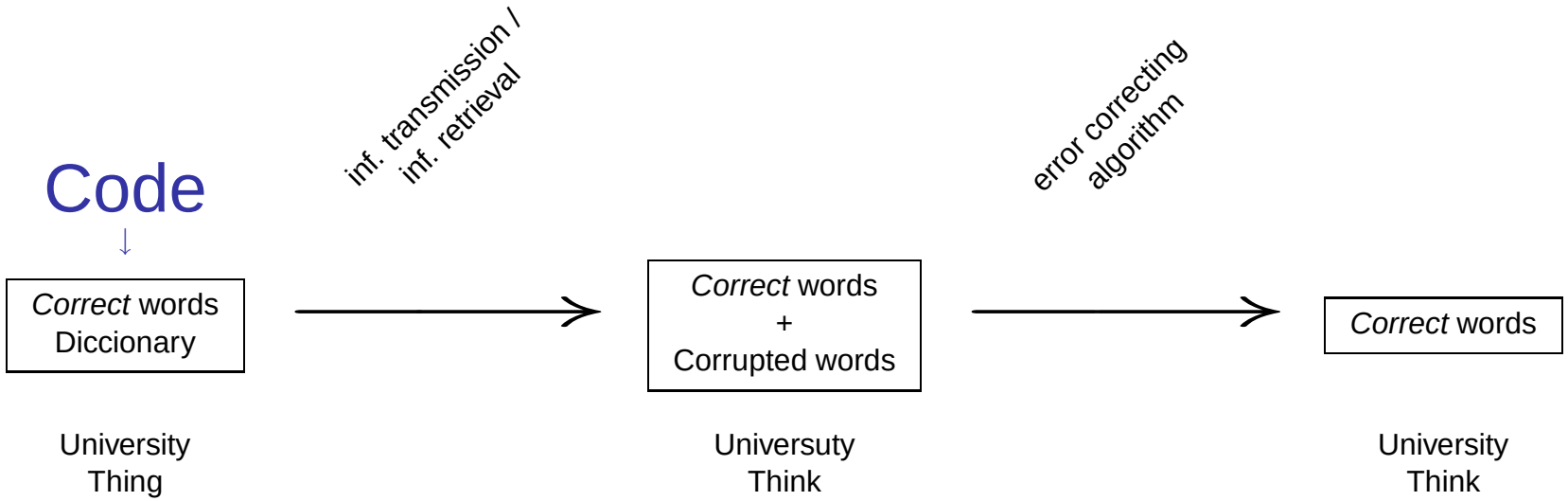
Let Σ be an alphabet.
Let $x \in \Sigma^n$ be a database.
Suppose x is replicated among one or more servers.

A PIR protocol allows a user to retrieve x_i from the server(s) while hiding i.

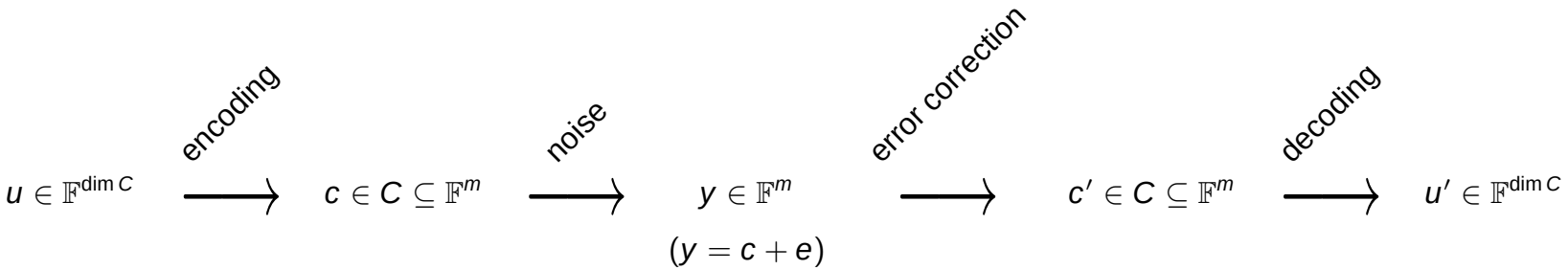
- Information-theoretic PIR:
  - more than one server,
  - privacy refers to each individual server.
- Computational PIR:
  - one server which is computationally bounded,
  - privacy is relaxed to *computational* privacy.

Codes and errors



Linear codes

A linear code of length m is a vector subspace C of $\mathbb{F}^m$.



Usual error correction approach:

obtain u' from y.

Locally decoding approach:

obtain the bit u'_i from a (small) number of randomly chosen bits in y.

Example

The Hamming code $H_2(7,4)$ has parity check matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Its dual code has encoding function

$$\begin{array}{rcl} E : \mathbb{Z}_2^3 & \to & \mathbb{Z}_2^7 \\ (x_1, x_2, x_3) & \mapsto & (x_1, x_2, x_3, x_1 + x_2, x_1 + x_3, x_2 + x_3, x_1 + x_2 + x_3) \end{array}$$

$$(\; x_1, \; x_2, \; x_3 \;) \mapsto (\; x_1, \; x_2, \; x_3, \; x_1 + x_2, \; x_1 + x_3, \; x_2 + x_3, \; x_1 + x_2 + x_3 \;)$$

Probabilistic locally decoding algorithm D

Input: $y \in \mathbb{Z}_2^7$
$i \in \{1, 2, 3\}$

- Randomly take $j_1 \in \{1, \ldots, 7\}$.
- If $j_1 \neq i$
  - Take the only j_2 such that $E(x)_{j_1} + E(x)_{j_2} = x_i$.
  - Output $y_{j_1} + y_{j_2}$.
- If $j_1 = i$
  - Output y_{j_1}.

$$\begin{array}{ccccccccccccccc} (& x_1, & x_2, & x_3 &) & \mapsto & (& x_1, & x_2, & x_3, & x_1+x_2, & x_1+x_3, & x_2+x_3, & x_1+x_2+x_3 &) \\ (& 1, & 0, & 1 &) & \mapsto & (& 1, & 0, & 1, & 1, & 0, & 1, & 0 &) \end{array}$$

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Probabilistic locally decoding algorithm D

Input: $y \in \mathbb{Z}_2^7$ $\qquad (1,0,1,1,0,1,0)$
$i \in \{1,2,3\}$ $\qquad 1$

- Randomly take $j_1 \in \{1,\ldots,7\}$.
- If $j_1 \neq i$
  - Take the only j_2 such that $E(x)_{j_1} + E(x)_{j_2} = x_i$.
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Probabilistic locally decoding algorithm D

Input: $y \in \mathbb{Z}_2^7$ $(1,0,1,1,0,1,0)$
$i \in \{1,2,3\}$ 1

- Randomly take $j_1 \in \{1,\ldots,7\}$. 4
- If $j_1 \neq i$
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Input: $y \in \mathbb{Z}_2^7$ $(1,0,1,1,0,1,0)$
$i \in \{1,2,3\}$ 1

- Randomly take $j_1 \in \{1,\ldots,7\}$. 4
- If $j_1 \neq i$
  - Take the only j_2 such that $E(x)_{j_1} + E(x)_{j_2} = x_i$. 2
  - Output $y_{j_1} + y_{j_2}$. $y_4 + y_2 = 1$
- If $j_1 = i$
  - Output y_{j_1}.

$$\begin{array}{ccccccccccccccc} (& x_1, & x_2, & x_3 &) & \mapsto & (& x_1, & x_2, & x_3, & x_1+x_2, & x_1+x_3, & x_2+x_3, & x_1+x_2+x_3 &) \\ (& 1, & 0, & 1 &) & \mapsto & (& 1, & 0, & 1, & 1, & 0, & 1, & 0 &) \end{array}$$

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- If $j_1 = i$
  - Output y_{j_1}. $y_1 = 1$

$$(\; x_1, \; x_2, \; x_3 \;) \mapsto (\; x_1, \; x_2, \; x_3, \; x_1 + x_2, \; x_1 + x_3, \; x_2 + x_3, \; x_1 + x_2 + x_3 \;)$$

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  - Output $y_{j_1} + y_{j_2}$.
- If $j_1 = i$
  - Output y_{j_1}.

Algorithm D reads at most 2 indices of y.

If $d(y, E(x)) = 0$ then for all i

$$Pr[D(y, i) = x_i] = 1.$$

If $d(y, E(x)) = 1$??
If $i = 1$ and $y - E(x) = 1000000$ then

$$Pr[D(y, i) = x_i] = \frac{6}{7}.$$

The same happens if $i = 2$ and $y - E(x) = 0100000$ or $i = 3$ and $y - E(x) = 0010000$.

In the remaining cases

$$Pr[D(y, i) = x_i] = \frac{5}{7}.$$

Definition (Katz, Trevisan, 2000)

For fixed $\delta, \epsilon \in \mathbb{R}$ and $q \in \mathbb{Z}^+$, we say that

$$E : \{0,1\}^n \to \Sigma^m$$

is a (q, δ, ϵ)-locally decodable encoding if there exists a probabilistic algorithm D such that

- For every $x \in \{0,1\}^n$, for every $y \in \Sigma^m$, with $d(y, E(x)) \leqslant \delta m$ and for all $i \in [n]$, we have

$$Pr[D(y,i) = x_i] \geqslant \frac{1}{2} + \epsilon.$$

- In every invocation, D reads at most q indices of y.

Example

In the previous example, E is a $(q = 2, \epsilon = \frac{5}{7} - \frac{1}{2} = \frac{3}{14}, \delta = \frac{1}{7})$-locally decodable encoding.

Example (trivial)

A linear code with dimension k and correction capability $t = \lfloor \frac{d-1}{2} \rfloor$ is

- $(m, t/m, 1/2)$-locally decodable code,
- $(k, 0, 1/2)$-locally decodable code.

Shortcoming

Suppose $Pr[D(\cdot, i) \text{ reads index } j] \approx 1$.
An adversary can corrupt $E(x)_j$ affecting the performance of D.

In the previous example,

$$Pr[D(\cdot, i) \text{ reads index } j] = \begin{cases} 1/7 & \text{if } i = j \\ 2/7 & \text{if } i \neq j \end{cases}$$

Definition (Katz, Trevisan, 2000)

For fixed $c, \epsilon \in \mathbb{R}$ and $q \in \mathbb{Z}^+$, we say that

$$E : \{0,1\}^n \to \Sigma^m$$

is a (q, c, ϵ)-smooth encoding if there exists a probabilistic algorithm D such that

- For every $x \in \{0,1\}^n$ and for all $i \in [n]$, we have

$$Pr[D(E(x), i) = x_i] \geqslant \frac{1}{2} + \epsilon.$$

- In every invocation, D reads at most q indices of y.
- For every $i \in [n]$ and $j \in [m]$, we have:

$$Pr[D(\cdot, i) \text{ reads index } j] \leqslant \frac{c}{m}.$$

Example

The encoding of the previous example is a $(q = 2, c = 2, \epsilon = \frac{3}{14})$-smooth encoding.

Example (trivial)

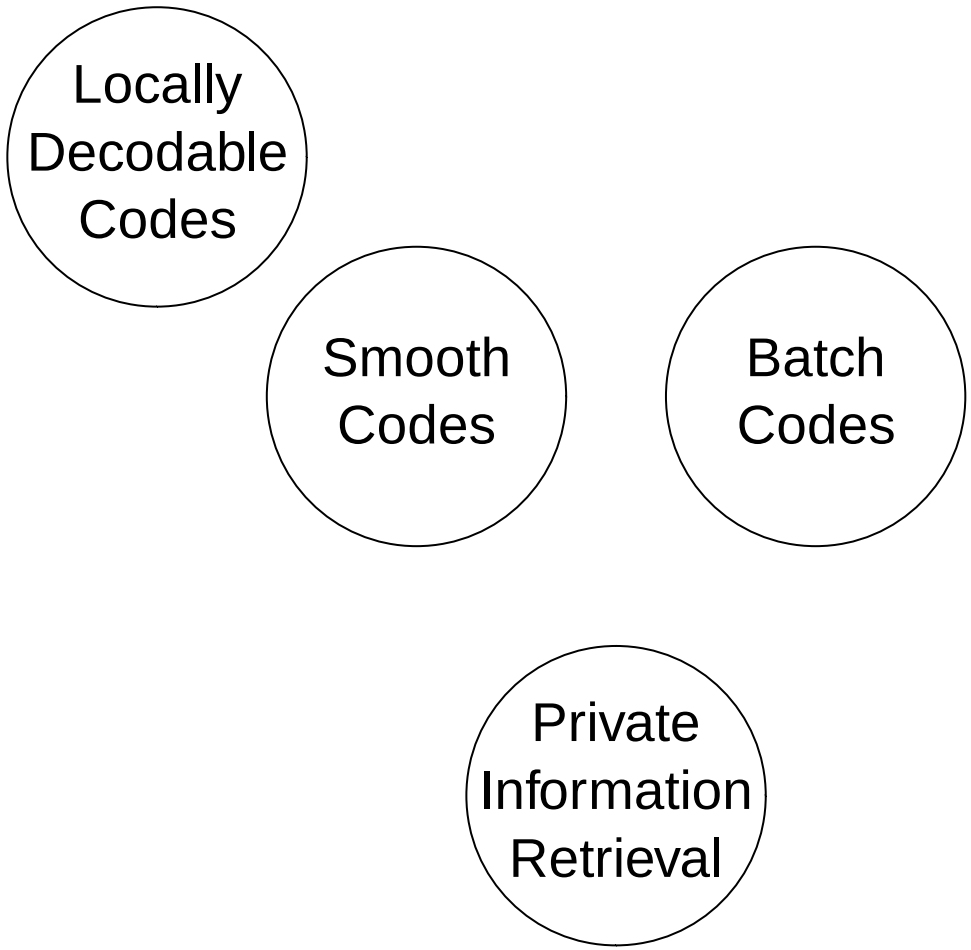
A linear code in $\mathbb{F}^m$ is a $(m, m, 1/2)$-locally decodable code.

Remark

- Any (q, c, ϵ)-smooth encoding is a $(q, 0, \epsilon)$-locally decodable encoding and so, it is a (q, δ, ϵ)-locally decodable encoding for some $\delta \geqslant 0$.
- Any (q, δ, ϵ)-locally decodable encoding is a (q, m, ϵ)-smooth encoding and so it is a (q, c, ϵ)-smooth encoding for some $c \leqslant m$.

Theorem (Katz, Trevisan, 2000)

Any (q, δ, ϵ)-locally decodable encoding is a $(q, q/\delta, \epsilon)$-smooth encoding.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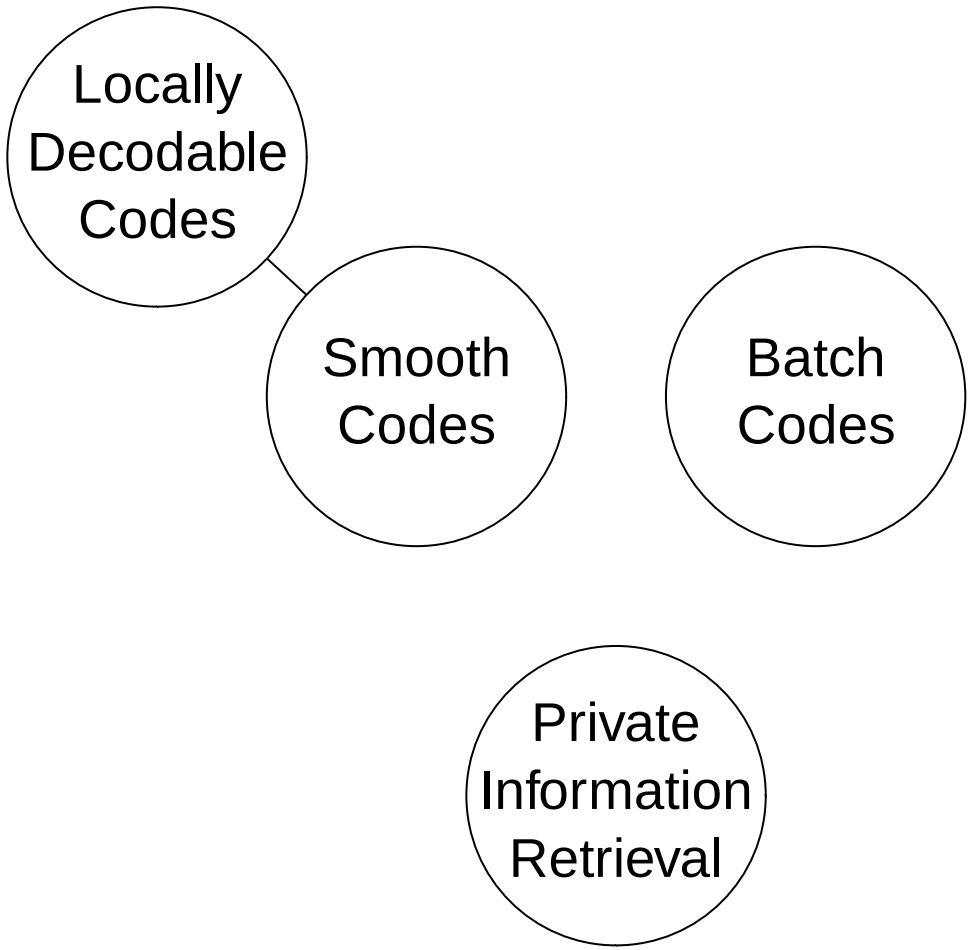
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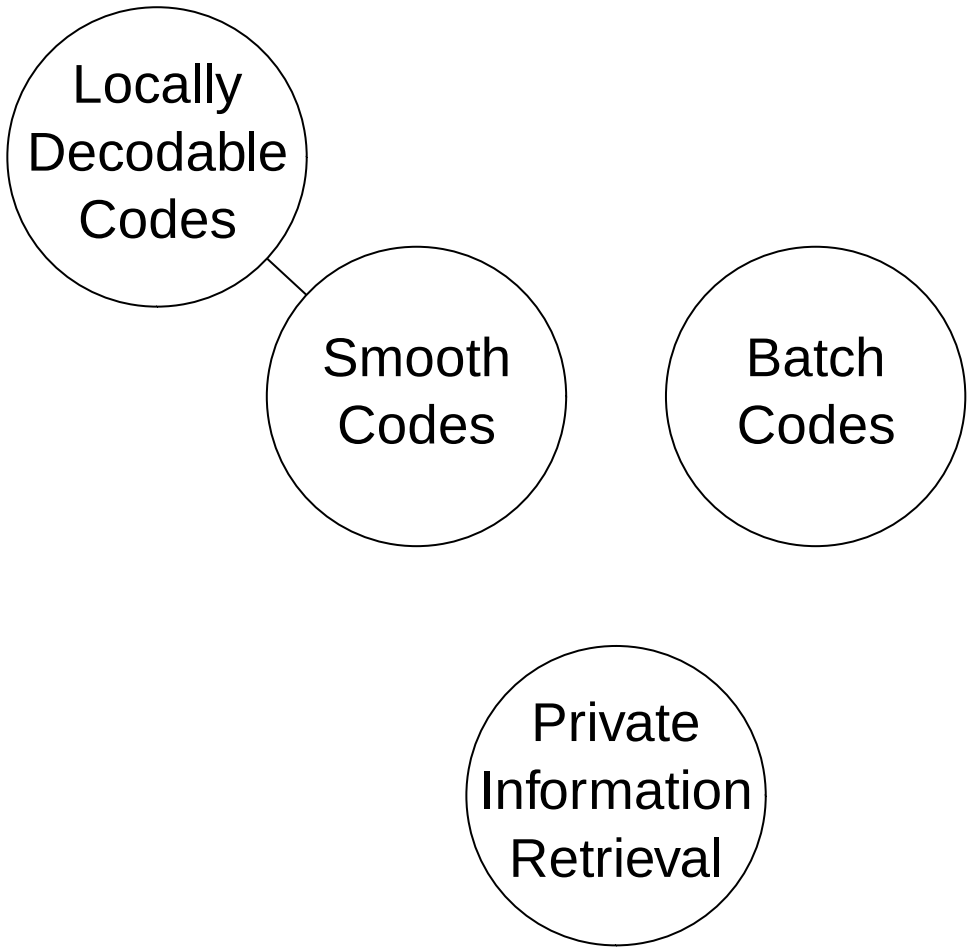
Results from Katz, Trevisan, 2000

Any (q, c, ϵ)-smooth decoding algorithm (with $c > q$) can be converted into a $(q, q, \epsilon^2/2c)$-smooth decoding algorithm with uniformly distributed queries.
A (q, q, ϵ)-smooth encoding $C : \{0,1\}^n \to \Sigma^m$ with uniformly distributed queries gives a information-theoretic PIR scheme with q servers, query size $\log m$, and answer size $\log |\Sigma|$, such that the user has probability $1/2 + \epsilon$ of correctly retrieving the bit she is interested in.

A PIR scheme with q servers, query size t, answer size l, and probability of correct retrieval $1/2 + \epsilon$ gives a $(q, q, \epsilon/2)$-smooth encoding $C : \{0,1\} \to (\{0,1\}^l)^m$ where $m = O(q2^t/\epsilon)$.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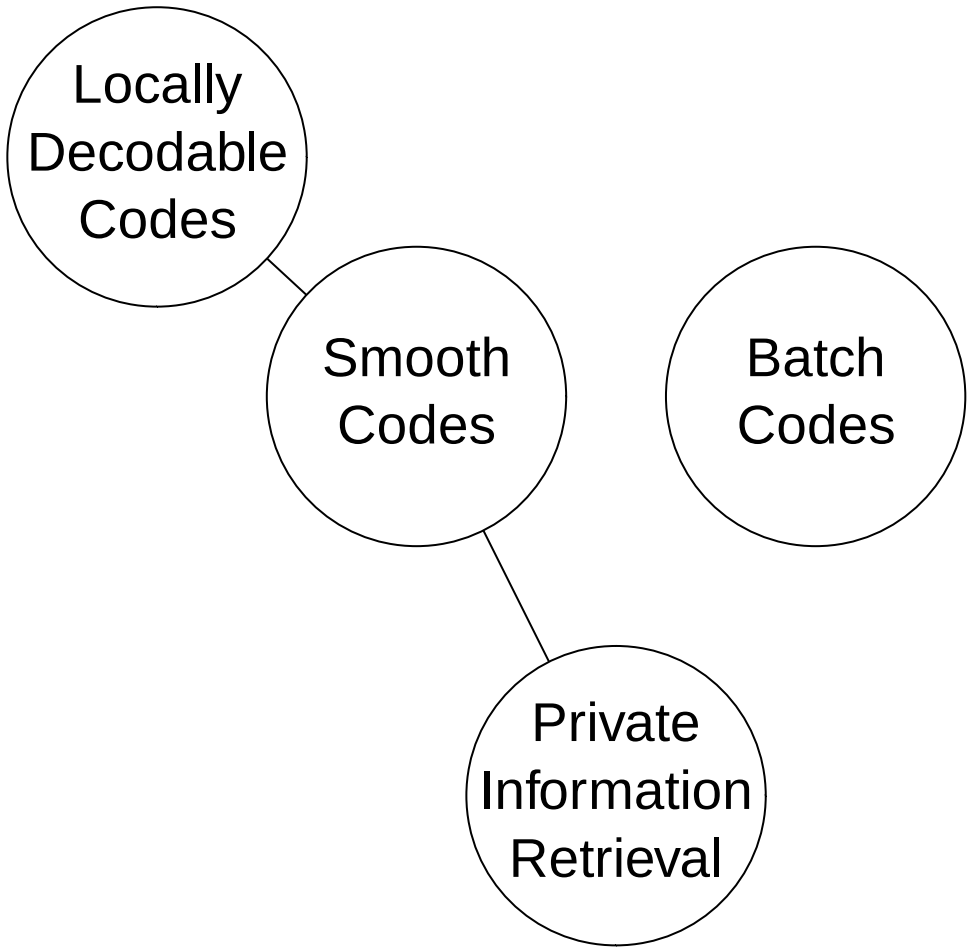
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Motivation

Suppose we want to store the string

$$\overbrace{10110100110101}^{14 bits}$$

in devices that can contain only 7 bits.

Option 1.

Use two devices,

$$E(\overbrace{1011010}^{L}\overbrace{0110101}^{R}) = (\overbrace{1011010}^{L}, \overbrace{0110101}^{R})$$

To retrieve 2 bits from the string we may need to retrieve 2 bits from the same device.

Option 2.

Use three devices,

$$E(\overbrace{1011010}^{L}\overbrace{0110101}^{R}) = (\overbrace{1011010}^{L}, \overbrace{0110101}^{R}, \overbrace{1101111}^{L\oplus R})$$

We can retrieve 2 bits from the string reading 1 bit from each device:

$$D_{3,11}(10\mathbf{1}1010\,,\, 011\mathbf{0}0101\,,\, 1101111) = (1,0)$$

$$D_{3,4}(10\mathbf{1}1010\,,\, 011\mathbf{0}0101\,,\, 110\mathbf{1}111) = (1,1)$$

$$D_{10,11}(10\mathbf{1}1010\,,\, 011\mathbf{0}0101\,,\, 11\mathbf{0}1111) = (1,0)$$

Definition (Ishai, Kushilevitz, Ostrovsky, Sahai, 2004)

A (n, N, k, m) batch code over Σ is an encoding function

$$E : \Sigma^n \to (\Sigma^*)^m$$

together with a decoding function (algorithm)

$$D : (\Sigma^*)^m \times \{1, \ldots, n\}^k \to \Sigma^k$$

such that

- $D(E(x), i_1, \ldots, i_k) = (x_{i_1}, \ldots, x_{i_k})$ for all $x \in \Sigma^n$.
- D reads at most 1 symbol from each of the m components of $E(x)$.

The m components of $E(x)$ are called *buckets*.

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- $D(E(x), i_1, \ldots, i_k) = (x_{i_1}, \ldots, x_{i_k})$ for all $x \in \Sigma^n$.
- D reads at most 1 symbol from each of the m components of $E(x)$.

The m components of $E(x)$ are called *buckets*.

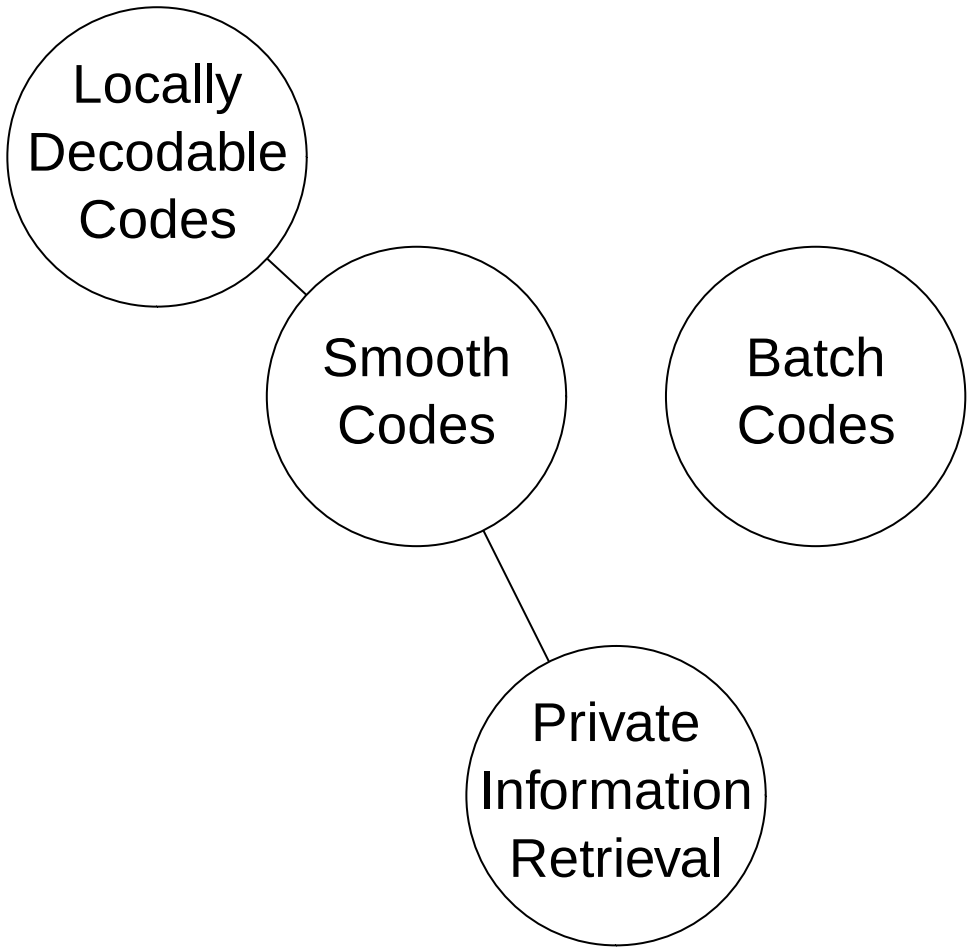
In the previous example $k = 2$, $N = \frac{3}{2}n$, $m = 3$.

A (n, N, k, m) multiset batch code is a (n, N, k, m) batch code such that for each *k-multi-query* $(i_1, \ldots, i_k)$ the set of m buckets admits a partition of k subsets of buckets such that the i_jth single query can be recovered by reading at most one symbol from each of the buckets in the i_jth subset.

Theorem (Ishai, Kushilevitz, Ostrovsky, Sahai, 2004)

Let $C : \Sigma^n \to \Sigma^m$ be a q-query smooth encoding. Then C describes a $(n, m, \lfloor \frac{m}{q^2} \rfloor, m)$-multiset batch code.

Conversely, the decoding procedure of a (n, m, k, m) multiset batch code gives an *expected* (m/k)-query smooth decoding procedure:
to smoothly decode x_i, run the batch decoder on the multiset $\{i, i, \ldots, i\}$, and pick a random set of buckets from the k disjoint sets allowing to decode x_i.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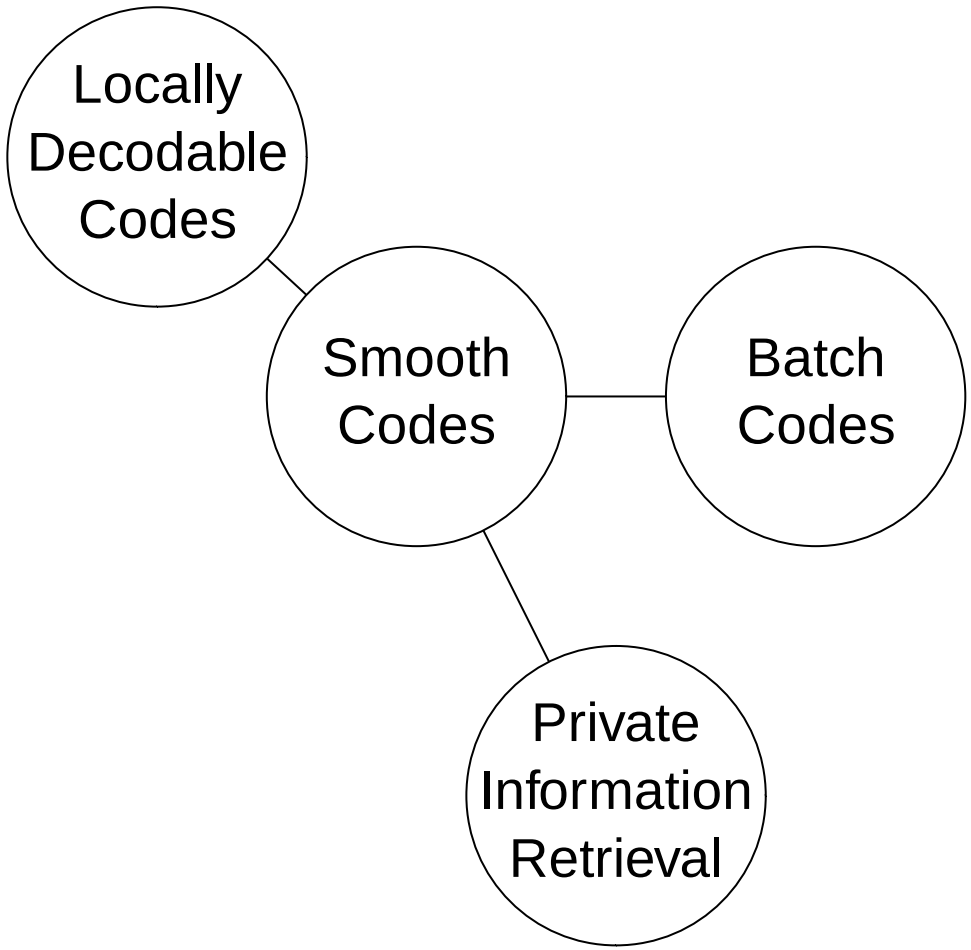
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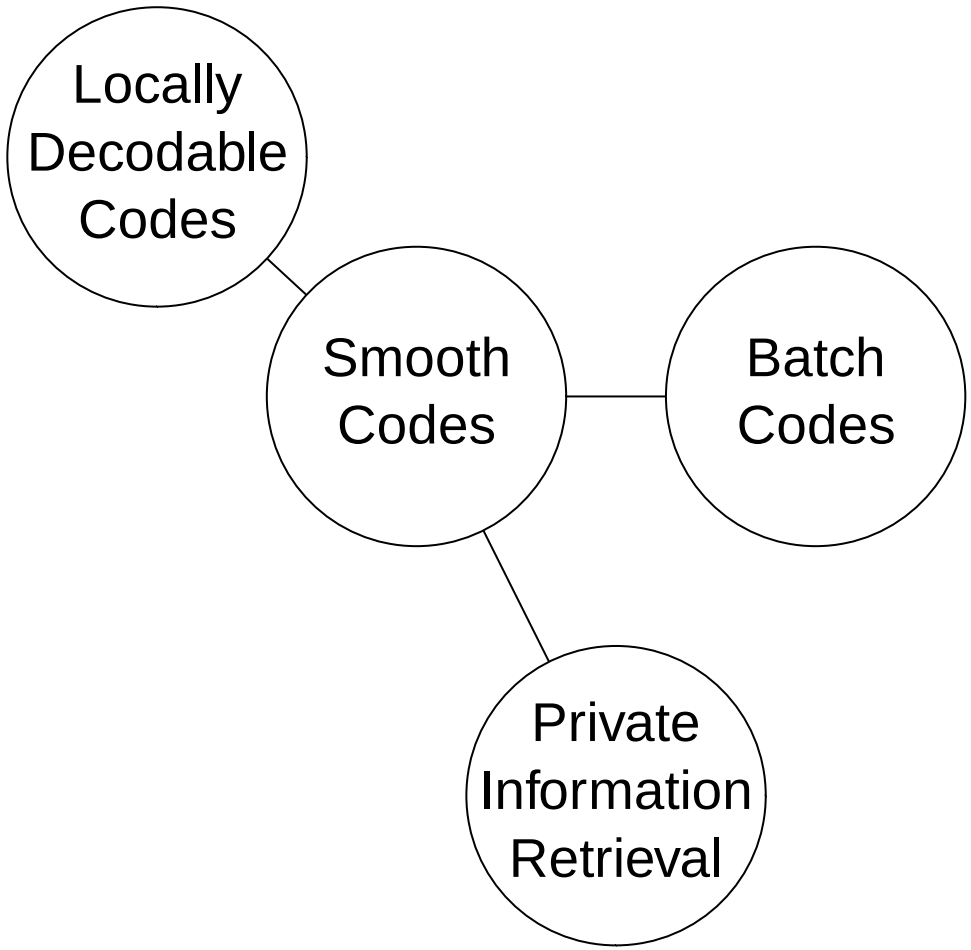
A (n, N, k, m) batch code with bucket sizes N_j, $1 \leqslant k \leqslant m$, provides a reduction from k-query PIR to m invocations of standard PIR on databases of size N_j.

Any nontrivial batch code satisfying $N \ll nk$ implies amortized savings to the time complexity.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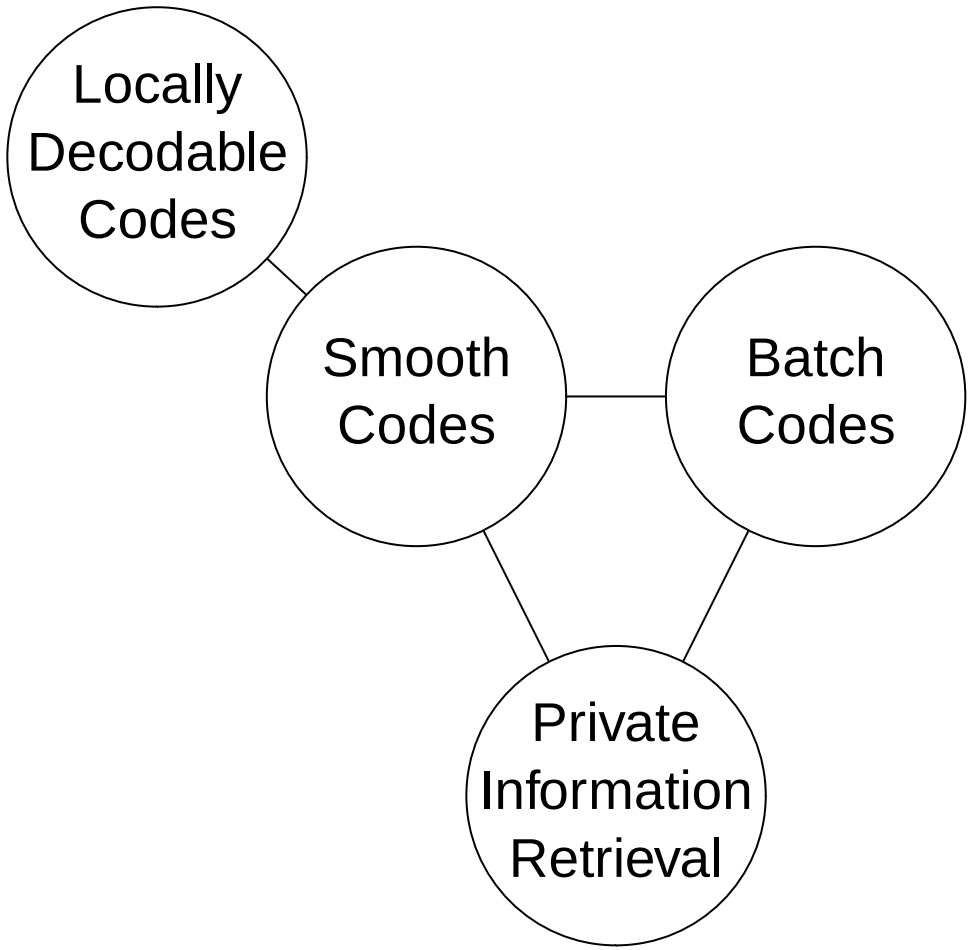
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Ongoing research topics

- k-PIR
- Peer-to-peer PIR
- Private file sharing system with ECC

What if the database does not cooperate?

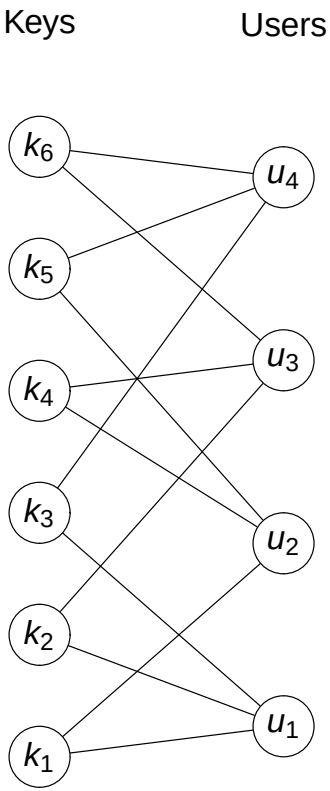
k-PIR: User can mask her target query by ORing it with $k-1$ fake queries.

For maximum privacy,

frequency(target query)=frequency(fake queries).

What if we don’t know the frequencies of all queries?

A configuration is a bipartite graph with constant degree in both sets of vertices and with no cycle of length 4.



A peer-to-peer PIR system can be defined based on the graph.

Private file sharing
needs error correction

n s d f g h j k l c
p h e s c t d x e s
o l p a x r s z g a
i k q l z e a u w p
u j m p d w s l q o
y h n o k k p k m i
t g b i e q j j n u
r f v o h m o b b y
e d y y g n i h u t
w j x t f b u g v g
q a z r d v y f c r

Private file sharing
needs error correction

C	s	d	f	g	h	j	k	l	c
p	**O**	e	s	c	t	d	x	e	s
o	l	**N**	a	x	r	s	z	g	a
i	k	q	**S**	z	e	a	u	w	p
u	j	m	p	**O**	w	s	l	q	o
y	h	n	o	k	**L**	p	k	m	i
t	g	b	i	e	q	**I**	j	n	u
r	f	v	o	h	m	o	**D**	b	y
e	d	y	y	g	n	i	h	**E**	t
w	j	x	t	f	b	u	g	v	**R**
q	a	z	r	d	v	y	f	c	r

Private file sharing
needs error correction

C	s	d	f	g	h	j	k	l	c
p	**O**	e	s	c	t	d	x	e	s
o	l	**N**	a	**C**	**A**	**S**	**T**	**R**	**O**
i	k	q	**S**	z	e	a	u	w	p
u	j	m	p	**O**	w	s	l	q	o
y	h	n	o	k	**L**	p	k	m	i
t	g	b	i	e	q	**I**	j	n	u
r	f	v	o	h	m	o	**D**	b	y
e	d	y	y	g	n	i	h	**E**	t
w	j	x	t	f	b	u	g	v	**R**
q	a	z	r	d	v	y	f	c	r

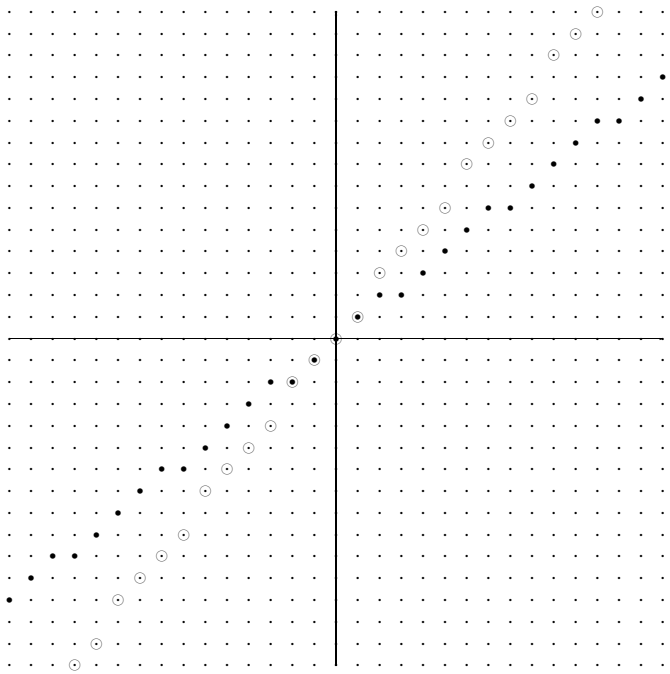
Private file sharing
needs error correction

C	s	d	f	g	h	j	k	l	c
p	**O**	e	s	c	t	d	x	e	s
o	l	**N**	a	**C**	**A**	**S**	**T**	**S**	**O**
i	k	q	**S**	z	e	a	**E**	w	p
u	j	m	p	**O**	w	**L**	l	q	o
y	h	n	o	k	**A**	p	k	m	i
t	g	b	i	**I**	q	**I**	j	n	u
r	f	v	**D**	h	m	o	**D**	b	y
e	d	**R**	y	g	n	i	h	**E**	t
w	**U**	x	t	f	b	u	g	v	**R**
q	a	z	r	d	v	y	f	c	r

Private file sharing needs error correction

Problems!

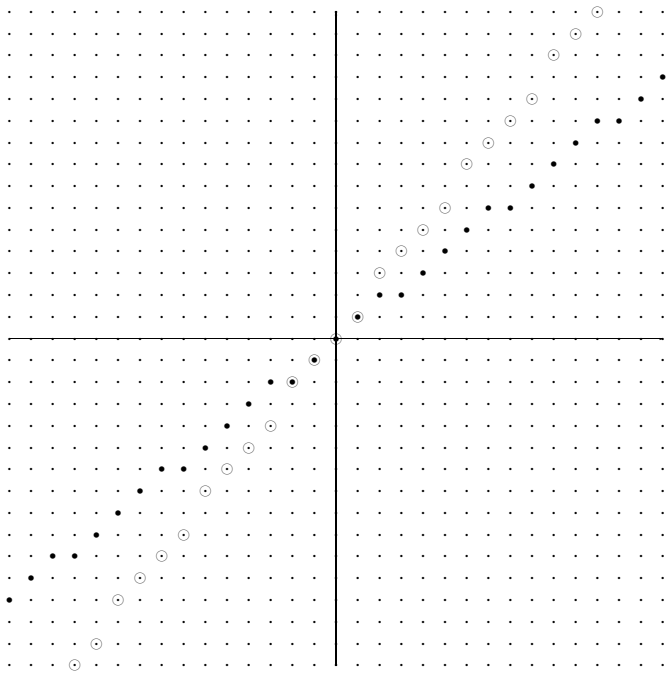
C	s	d	f	g	h	j	k	l	c
p	**O**	e	s	c	t	d	x	e	s
o	l	**N**	a	**C**	**A**	**S**	**T**	**S**	**O**
i	k	q	**S**	z	e	a	**E**	w	p
u	j	m	p	**O**	w	**L**	l	q	o
y	h	n	o	k	**A**	p	k	m	i
t	g	b	i	**I**	q	**I**	j	n	u
r	f	v	**D**	h	m	o	**D**	b	y
e	d	**R**	y	g	n	i	h	**E**	t
w	**U**	x	t	f	b	u	g	v	**R**
q	a	z	r	d	v	y	f	c	r



Private file sharing needs error correction

Problems!

C	s	d	f	g	h	j	k	l	c
p	**O**	e	s	c	t	d	x	e	s
o	l	**N**	a	**C**	**A**	**S**	**T**	**S**	**O**
i	k	q	**S**	z	e	a	**E**	w	p
u	j	m	p	**O**	w	**L**	l	q	o
y	h	n	o	k	**A**	p	k	m	i
t	g	b	i	**I**	q	**I**	j	n	u
r	f	v	**D**	h	m	o	**D**	b	y
e	d	**R**	y	g	n	i	h	**E**	t
w	**U**	x	t	f	b	u	g	v	**R**
q	a	z	r	d	v	y	f	c	r



Alternative: combinatorial structures